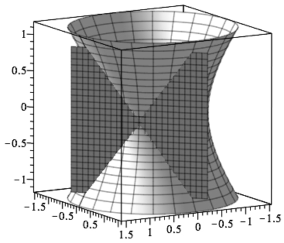
1. $\boxed{\mathbf{r}(u,v)=(2u,u^2+v,v^2)}$. a) $\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2u&0\\0&1&2v\end{vmatrix}=2(2uv,-2v,1)\xrightarrow{u=0,\,v=1}2(0,-2,1)$.

Plano tangente: $(0,-2,1)\cdot(x,y-1,z-1)=0\;\to\;z=2y-1$.

b) $u=\frac{x}{2}$, $v=y-\frac{x^2}{4}$, $z=(y-\frac{x^2}{4})^2$. $z_x(0,1)=0$, $z_y(0,1)=2$, $\nwarrow$ $z=1+0(x-1)+2(x-1)$.

2. $\boxed{\mathbf{r}(u,v)=(\operatorname{ch}u\cos v,\operatorname{ch}u\operatorname{sen}v,\operatorname{sh}u)}$ $\begin{matrix}u\in\mathbf{R}\\v\in[0,2\pi]\end{matrix}$ $\quad x^2+y^2-z^2=\operatorname{ch}^2u-\operatorname{sh}^2u=1$.
$x^2+y^2=\operatorname{ch}^2u$ curvas de nivel.

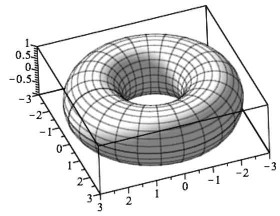


$\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\\operatorname{sh}u\cos v&\operatorname{sh}u\operatorname{sen}v&\operatorname{ch}u\\-\operatorname{ch}u\operatorname{sen}v&\operatorname{ch}u\cos v&0\end{vmatrix}=\operatorname{ch}u(-\operatorname{ch}u\cos v,-\operatorname{ch}u\operatorname{sen}v,\operatorname{sh}u)\xrightarrow{u=0,\,v=\pi/4}-\Big(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\Big)$.

$\mathbf{r}(0,\frac{\pi}{4})=(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0)$. Plano tangente: $-\Big(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\Big)\cdot\Big(x-\frac{1}{\sqrt2},y-\frac{1}{\sqrt2},z\Big)=0$, $x+y=\sqrt2$.

Partiendo de $F(x,y,z)=x^2+y^2-z^2=1$. $\nabla F=(2x,2y,-2z)\big|_{(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0)}=(\sqrt2,\sqrt2,0)^{\uparrow}$

3. $\boxed{\mathbf{r}(\theta,\phi)=((2+\cos\phi)\cos\theta,(2+\cos\phi)\operatorname{sen}\theta,\operatorname{sen}\phi)}$, $\theta,\phi\in[0,2\pi]$.



$\mathbf{r}_\theta\times\mathbf{r}_\phi=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-(2+\cos\phi)\operatorname{sen}\theta&(2+\cos\phi)\cos\theta&0\\-\operatorname{sen}\phi\cos\theta&-\operatorname{sen}\phi\operatorname{sen}\theta&\cos\phi\end{vmatrix}=(2+\cos\phi)\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-\operatorname{sen}\theta&\cos\theta&0\\-\operatorname{sen}\phi\cos\theta&-\operatorname{sen}\phi\operatorname{sen}\theta&\cos\phi\end{vmatrix}$,

$=(2+\cos\phi)(\cos\theta\cos\phi,\operatorname{sen}\theta\cos\phi,\operatorname{sen}\phi)$. $\|\mathbf{r}_\theta\times\mathbf{r}_\phi\|=2+\cos\phi$.

Área $=\iint_S dS=\iint_D\|\mathbf{r}_\theta\times\mathbf{r}_\phi\|\,d\theta\,d\phi=\int_0^{2\pi}\!\!\int_0^{2\pi}(2+\cos\phi)\,d\theta\,d\phi=8\pi^2$.

4. $\boxed{\iint_S(x^2+y^2)\,dS}$, $x^2+y^2+z^2=4\;\to\;\mathbf{r}(u,v)=(2\operatorname{sen}u\cos v,2\operatorname{sen}u\operatorname{sen}v,2\cos u)$, $\|\mathbf{r}_u\times\mathbf{r}_v\|=4\operatorname{sen}u$

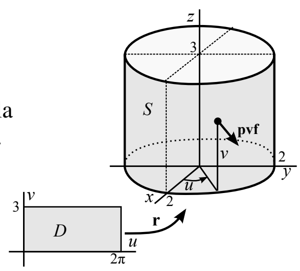
$\iint_S(x^2+y^2)\,dS=16\int_0^{2\pi}\!\!\int_0^{\pi}\operatorname{sen}^3u\,du\,dv=32\pi\int_0^{\pi}\operatorname{sen}u(1-\cos^2u)\,du=32\pi\big[\frac13\cos^3u-\cos u\big]_0^{\pi}=\frac{128}{3}\pi$.

Con $\mathbf{r}(x,y)=(x,y,\sqrt{4-x^2-y^2})$ [integrando y recinto simétricos en z , basta esta y multiplicar por 2 la integral].

$\|\mathbf{r}_x\times\mathbf{r}_y\|=\sqrt{(f_x)^2+(f_y)^2+1}=\Big[\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1\Big]^{1/2}=\frac{2}{\sqrt{4-x^2-y^2}}\;\Rightarrow$

$2\iint_S(x^2+y^2)\,dS=4\iint_B\frac{x^2+y^2}{\sqrt{4-x^2-y^2}}\,dx\,dy\underset{\text{polares}\ \uparrow}{=}4\int_0^{2\pi}\!\!\int_0^2\frac{r^3}{\sqrt{4-r^2}}\,dr\,d\theta=4\pi\int_0^4\frac{s\,ds}{\sqrt{4-s}}=\frac{128}{3}\pi$.

5. Parametrización de $x^2+y^2=4$: $\left.\begin{matrix}x=2\cos u\\y=2\operatorname{sen}u\\z=v\end{matrix}\right\}$ $\begin{matrix}u\in[0,2\pi]\\v\in[0,3]\end{matrix}$ $\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2s&2c&0\\0&0&1\end{vmatrix}=2(\cos u,\operatorname{sen}u,0)$.



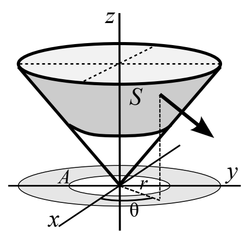
a) área de $S=\iint_S 1\,dS=\iint_D\|\mathbf{r}_u\times\mathbf{r}_v\|\,du\,dv=\int_0^{2\pi}\!\!\int_0^3 2\,dv\,du=12\pi$ [claro, longitud de la base por la altura].

b) i) $\boxed{f(x,y,z)=x^2}$ $\iint_S f\,dS=\int_0^{2\pi}\!\!\int_0^3 8\cos^2u\,dv\,du=12\int_0^{2\pi}(1+\cos2v)\,dv=24\pi$.

ii) $\boxed{\mathbf{f}(x,y,z)=(xz,yz,2)}$ $\mathbf{f}(\mathbf{r}(u,v))=2(v\cos u,v\operatorname{sen}u,1)$.

$\iint_S\mathbf{f}\cdot d\mathbf{S}=\iint_D\mathbf{f}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv=\int_0^{2\pi}\!\!\int_0^3 4v\,dv\,du=36\pi$.

6. $\boxed{z^2=x^2+y^2}$. a) $\left.\begin{matrix}x=r\cos\theta\\y=r\operatorname{sen}\theta\\z=r\end{matrix}\right\}$ $\begin{matrix}r\in[1,2]\\\theta\in[0,2\pi]\end{matrix}$ $\mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\c&s&1\\-rs&rc&0\end{vmatrix}=(-r\cos\theta,-r\operatorname{sen}\theta,r)$.
[apunta hacia interior]



$\iint_S\mathbf{f}\cdot d\mathbf{S}=-\iint_D(r\cos\theta,r\operatorname{sen}\theta,1)\cdot(-r\cos\theta,-r\operatorname{sen}\theta,r)\,dr\,d\theta=$ [$\mathbf{f}(x,y,z)=(x,y,1)$]

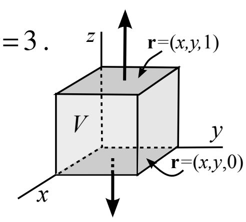
$=\int_0^{2\pi}\!\!\int_1^2[r^2-r]\,dr\,d\theta=2\pi\big[\frac{r^3}{3}-\frac{r^2}{2}\big]_1^2=\boxed{\frac53\pi}$. [$\mathbf{f}$ apunta también hacia exterior]

De otra forma: $\mathbf{r}(x,y)=(x,y,\sqrt{x^2+y^2})$, $\mathbf{r}_x\times\mathbf{r}_y=(-x(x^2+y^2)^{-1/2},-y(x^2+y^2)^{-1/2},1)$.

$-\iint_A(x,y,1)\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dx\,dy=-\iint_A\big[1-\sqrt{x^2+y^2}\,\big]\,dx\,dy=\int_0^{2\pi}\!\!\int_1^2[r^2-r]\,dr\,d\theta=\cdots$ como antes.

b) $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\\partial/\partial x&\partial/\partial y&\partial/\partial z\\x&y&1\end{vmatrix}=(0,0,0)$ y $\mathbf{f}\in C^1(\mathbf{R}^2)\;\Rightarrow\;\mathbf{f}$ es conservativo. [De hecho la $U=xy+z$ se ve a ojo].
Su integral sobre toda línea cerrada es 0 , sobre esa circunferencia en particular.

7. $\boxed{\mathbf{f}(x,y,z)=(x^2,y^2,z^2)}$ $\iiint_V \operatorname{div}\mathbf{f}=\int_0^1\!\int_0^1\!\int_0^1(2x+2y+2z)\,dx\,dy\,dz=\int_0^1 2x\,dx+\int_0^1 2y\,dy+\int_0^1 2z\,dz=3$.



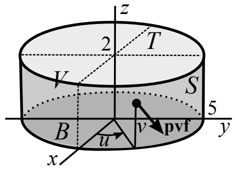
$\iint_{\partial V}\mathbf{f}\cdot\mathbf{n}\,dS=\iint+\iint+\iint+\iint+\iint+\iint$ sobre cada cara.

Para la de arriba: $\mathbf{r}=(x,y,1)$, $\mathbf{f}(\mathbf{r})=(x^2,y^2,1)$, $\mathbf{n}=(0,0,1)$, $\iint_{\square}1\,dS=1$.

Para la de abajo: $\mathbf{r}=(x,y,0)$, $\mathbf{f}(\mathbf{r})=(x^2,y^2,0)$, $\mathbf{n}=(0,0,-1)$, $\iint_{\square}0\,dS=0$.

Y las otras cuatro igual. $1+0+1+0+1+0=3$ como debía ser.

8. $\boxed{\mathbf{f}(x,y,z)=4x\,\mathbf{i}+4y\,\mathbf{j}+z^2\,\mathbf{k}}$ en el cilindro $x^2+y^2\le 25$, $0\le z\le 2$. $\operatorname{div}\mathbf{f}=8+2z$.



$\iiint_V \operatorname{div}\mathbf{f}=$ [cilíndricas] $=\int_0^{2\pi}\!\int_0^5\!\int_0^2 r(8+2z)\,dz\,dr\,d\theta=2\pi\cdot\frac{25}{2}(16+4)=\boxed{500\pi}$.

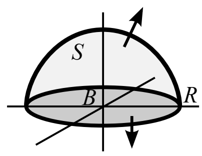
S dada por $\mathbf{r}=(5\cos\theta,5\operatorname{sen}\theta,z)$, $\mathbf{r}_\theta\times\mathbf{r}_z=5(\cos\theta,\operatorname{sen}\theta,0)$, $\theta\in[0,2\pi]$, $z\in[0,2]$.

Sobre ella: $\mathbf{f}(\mathbf{r}(\theta,z))=(20\cos\theta,20\operatorname{sen}\theta,z^2)$, $\iint_S\mathbf{f}\cdot d\mathbf{S}=\int_0^{2\pi}\!\int_0^2 100\,dz\,d\theta=400\pi$.

En la tapa superior T: $\mathbf{f}\cdot\mathbf{n}=(4x,4y,4)\cdot(0,0,1)=2\;\rightarrow\;\iint_{B_5}4=\pi\,5^2\,4=100\pi$.

En base B: $\mathbf{f}\cdot\mathbf{n}=(4x,4y,0)\cdot(0,0,-1)=0\;\rightarrow\;\iint_{B_5}0=0$. Por tanto, $\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}=400\pi+100\pi+0=\boxed{500\pi}$.

9. $\boxed{\mathbf{f}(x,y,z)=(y,-x,1)}$ Flujo $\Phi=\Phi_S+\Phi_B$ a través de la superficie cerrada $\partial V=S\cup B$ (hemisferio superior de la superficie esférica de radio R unido al círculo que es su base):



Parametrización $\mathbf{r}(\phi,\theta)$ de S: $\left.\begin{aligned}x&=R\operatorname{sen}\phi\cos\theta\\ y&=R\operatorname{sen}\phi\operatorname{sen}\theta\\ z&=R\cos\phi\end{aligned}\right\}$ $\begin{matrix}\phi\in[0,\frac{\pi}{2}]\\ \theta\in[0,2\pi]\end{matrix}$ $\begin{matrix}\mathbf{r}_\phi=(R\,c\phi\,c\theta,R\,c\phi\,s\theta,-R\,s\phi)\\ \mathbf{r}_\theta=(-R\,s\phi\,s\theta,R\,s\phi\,c\theta,0)\end{matrix}$

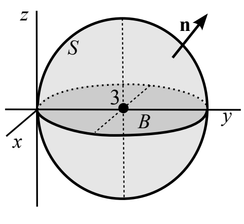
$$\mathbf{r}_\phi\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ R\,c\phi\,c\theta&R\,c\phi\,s\theta&-R\,s\phi\\ -R\,s\phi\,s\theta&R\,s\phi\,c\theta&0\end{vmatrix}=R^2(s^2\phi\,c\theta,s^2\phi\,s\theta,s\phi\,c\phi)=R^2\operatorname{sen}\phi\,(\operatorname{sen}\phi\cos\theta,\operatorname{sen}\phi\operatorname{sen}\theta,\cos\phi)$$
$=R\operatorname{sen}\phi\,\mathbf{r}(\phi,\theta)$ (apunta hacia afuera).

$$\Phi_S=\iint_S\mathbf{f}\cdot d\mathbf{S}=\int_0^{\pi/2}\!\int_0^{2\pi}R^2\operatorname{sen}\phi\cos\phi\,d\theta\,d\phi=2\pi R^2\big[\tfrac{1}{2}\operatorname{sen}^2\phi\big]_0^{\pi/2}=\pi R^2 .$$

Para B es $\mathbf{n}=(0,0,-1)$, $\mathbf{f}\cdot\mathbf{n}=-1\;\Rightarrow\;\Phi_B=\iint_B\mathbf{f}\cdot\mathbf{n}\,dS=-\iint_B dS=-\pi R^2$ [$=-$área]. $\Phi=\Phi_S+\Phi_B=\boxed{0}$.

Comprobamos el teorema de Gauss. Como $\nabla\cdot\mathbf{f}=0$: $\iiint_V\nabla\cdot\mathbf{f}\,dV=\iiint_V 0\,dV=\boxed{0}=\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}$.

10. $\boxed{\mathbf{f}(x,y,z)=3yz\,\mathbf{i}+2xz\,\mathbf{j}+(z+xy)\,\mathbf{k}}$ $x^2-6x+y^2+z^2=0\;\Leftrightarrow\;(x-3)^2+y^2+z^2=3^2$ [esfera de radio 3 centrada en $(3,0,0)$].



Como $\operatorname{div}\mathbf{f}=1$, el teorema de Gauss implica que el flujo pedido es:

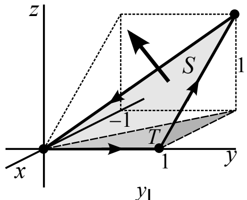
$\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}=\iiint_V 1\,dx\,dy\,dz=$ volumen de $V=\frac{4}{3}\pi 3^3=36\pi$.

Calcularlo directamente sería largo, tanto usando esféricas centradas en el punto:

$\mathbf{r}(\phi,\theta)=(3+3\operatorname{sen}\phi\cos\theta,3\operatorname{sen}\phi\operatorname{sen}\theta,3\cos\phi)$, $\phi\in[0,\pi]$, $\theta\in[0,2\pi]$,

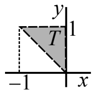
como las parametrizaciones cartesianas: $(x,y,\pm\sqrt{6x-x^2-y^2})$, $(x,y)\in B$ círculo de radio 3 centrado en $(3,0)$.

11. $\boxed{\mathbf{f}(x,y,z)=(yz,e^y,1)}$ $\rightarrow$ $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ yz&e^y&1\end{vmatrix}=(0,y,-z)$. vector normal: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 0&1&0\\ -1&1&1\end{vmatrix}=\mathbf{i}+\mathbf{k}$.



Triángulo sobre el plano $(1,0,1)\cdot(x,y,z)=x+z=0\;\rightarrow\;\mathbf{r}(x,y)=(x,y,-x)$ en T , $\overset{\uparrow}{\mathbf{r}_x\times\mathbf{r}_y}$

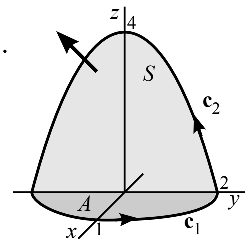
$\iint_S\operatorname{rot}\mathbf{f}\cdot\mathbf{n}\,dS=\int_{-1}^0\!\int_{-x}^1(0,y,x)\cdot(1,0,1)\,dy\,dx=\int_{-1}^0(x+x^2)\,dx=\big[\frac{x^2}{2}+\frac{x^3}{3}\big]_{-1}^0=\boxed{-\frac{1}{6}}$.



Parametrizamos ∂S:
$\mathbf{c}_1(t)=(0,t,0)$, $t\in[0,1]$ $\quad\mathbf{c}_1'=(0,1,0)\quad\mathbf{f}(\mathbf{c}_1)=(0,e^t,1)$
$\mathbf{c}_2(t)=(-t,1,t)$, $t\in[0,1]$ $\quad\mathbf{c}_2'=(-1,0,1)\quad\mathbf{f}(\mathbf{c}_2)=(t,e,1)\;\rightarrow$
$\mathbf{c}_3(t)=(-t,t,t)$, $t\in[1,0]$ $\quad\mathbf{c}_3'=(-1,1,1)\quad\mathbf{f}(\mathbf{c}_3)=(t^2,e^t,1)$

$$\oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_0^1 e^t\,dt+\int_0^1(1-t)\,dt+\int_1^0(1-t^2+e^t)\,dt=e-1+1-\tfrac{1}{2}-1+\tfrac{1}{3}+1-e=\boxed{-\tfrac{1}{6}}$$ como debía.

12. $\mathbf{f}(x,y,z)=(3,x^2,y)$, $\mathbf{r}(x,y)=(x,y,4-4x^2-y^2)$ $\to$ $\operatorname{rot}\mathbf{f}=(1,0,2x)$, $\mathbf{r}_x\times\mathbf{r}_y=(-f_x,-f_y,1)=(8x,2y,1)$.



$$\iint_S \operatorname{rot}\mathbf{f}\cdot d\mathbf{S}=\iint_A (1,0,2x)\cdot(8x,2y,1)\,dx\,dy=\int_{-2}^{2}\int_0^{\sqrt{1-y^2/4}}10x\,dx\,dy$$

$$=\int_{-2}^{2}5\left[1-\tfrac{1}{4}y^2\right]dy=20-\tfrac{5}{12}\left[y^3\right]_{-2}^{2}=\boxed{\tfrac{40}{3}}\,.$$

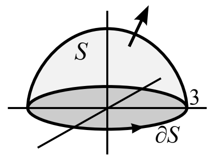
$\mathbf{c}_1(t)=(\cos t,2\operatorname{sen}t,0)$, $t\in[-\frac{\pi}{2},\frac{\pi}{2}]$ $\quad \mathbf{c}_1'=(-\operatorname{sen}t,2\cos t,0)$ $\quad \mathbf{f}(\mathbf{c}_1)=(3,\cos^2 t,2\operatorname{sen}t)$ $\to$

$\mathbf{c}_2(t)=(0,-t,4-t^2)$, $t\in[-2,2]$ $\quad \mathbf{c}_2'=(0,-1,-2t)$ $\quad \mathbf{f}(\mathbf{c}_2)=(3,0,-t)$

$$\oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_{-\pi/2}^{\pi/2}\big[2\cos t(1-\operatorname{sen}^2 t)-3\underbrace{\operatorname{sen}t}_{\text{impar}}\big]dt+\int_{-2}^{2}2t^2dt=2\left[2\operatorname{sen}t-\tfrac{2}{3}\operatorname{sen}^3 t\right]_0^{\pi/2}+\tfrac{32}{3}=\boxed{\tfrac{40}{3}}\,.$$

$\left[\text{Sale más largo parametrizando } \mathbf{c}_1(t)=\left((1-\tfrac{t^2}{4})^{1/2},t,0\right),\ t\in[-2,2]\right]$.

13. $\boxed{\mathbf{F}(x,y,z)=-y\,\mathbf{i}+2x\,\mathbf{j}+(x+z)\,\mathbf{k}}$. $\operatorname{rot}\mathbf{f}=(0,-1,3)$. Superficie $x^2+y^2+z^2=9$, $z\geq 0$:



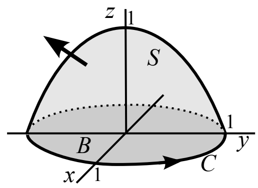
$\mathbf{r}(\phi,\theta)=(3\operatorname{sen}\phi\cos\theta,3\operatorname{sen}\phi\operatorname{sen}\theta,3\cos\phi)$, $\phi\in[0,\frac{\pi}{2}]$, $\theta\in[0,2\pi]$,

$\mathbf{r}_\phi\times\mathbf{r}_\theta=9(\operatorname{sen}^2\phi\cos\theta,\operatorname{sen}^2\phi\operatorname{sen}\theta,\operatorname{sen}\phi\cos\phi)$.

$$\iint_S \operatorname{rot}\mathbf{F}\cdot d\mathbf{S}=9\int_0^{2\pi}\int_0^{\pi/2}(3\operatorname{sen}\phi\cos\phi-\underbrace{\operatorname{sen}^2\phi\operatorname{sen}\theta}_{0\,\swarrow})\,d\phi\,d\theta=27\pi\left[\operatorname{sen}^2\phi\right]_0^{\pi/2}=\boxed{27\pi}\,.$$

Además: $\mathbf{c}(t)=(3\cos t,3\operatorname{sen}t,0)$, $t\in[0,2\pi]$ $\to \oint_{\partial S}\mathbf{F}\cdot d\mathbf{s}=9\int_0^{2\pi}\left[\operatorname{sen}^2 t+2\cos^2 t\right]dt=9\int_0^{2\pi}\frac{3+\cos t}{2}\,dt=\boxed{27\pi}$.

14. $\boxed{\mathbf{f}(x,y,z)=(x,xy,2z)}$ $\quad \operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ x&xy&2z\end{vmatrix}=(0,0,y)$. $\quad S$: $\mathbf{r}(x,y)=(x,y,1-x^2-y^2)$. $\mathbf{r}_x\times\mathbf{r}_y=(2x,2y,1)$.



$$\iint_S \operatorname{rot}\mathbf{f}\cdot d\mathbf{S}=\iint_B y\,dx\,dy=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}y\,dy\,dx=\boxed{0}$$ [o impar y recinto simétrico].

$\mathbf{c}(t)=(\cos t,\operatorname{sen}t,0)$, $t\in[0,2\pi]$ $\to \oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_0^{2\pi}(-\operatorname{sen}t\cos t+\cos^2 t\operatorname{sen}t)\,dt=\boxed{0}$.

$\operatorname{div}\mathbf{f}=3+x$. $\iiint_V \operatorname{div}\mathbf{f}=\text{[cilíndricas]}=\int_0^{2\pi}\int_0^1\int_0^{1-r^2}r(3+r\cos\theta)\,dz\,dr\,d\theta=6\pi\int_0^1(r-r^3)\,dr=\boxed{\tfrac{3}{2}\pi}$.

$$\iint_S \mathbf{f}\cdot d\mathbf{S}=\iint_B \underbrace{(x,xy,2-2x^2-2y^2)\cdot(2x,2y,1)}_{=2(1+xy-y^2)}\,dx\,dy=2\int_0^{2\pi}\int_0^1(r+r^3\cos\theta\operatorname{sen}\theta-r^3\operatorname{sen}^2\theta)\,dr\,d\theta=\tfrac{3}{2}\pi\,.$$

$\mathbf{r}_B(x,y)=(x,y,0)$, $(x,y)\in B$, $\iint_B \mathbf{f}\cdot\mathbf{n}\,dS=-\iint_B (x,xy,0)\cdot(0,0,-1)\,dx\,dy=0$. $\quad \iint_{S^*}=\iint_S+\iint_B=\boxed{\tfrac{3}{2}\pi}$.

15. a) $u\,\Delta u=u(u_{xx}+u_{yy})=\operatorname{div}(uu_x,uu_y)-(u_x^2+u_y^2)=\operatorname{div}(u\nabla u)-\|\nabla u\|^2$ (y casi igual para $n=3$).

b) $$\iint_D u\,\Delta u\,dx\,dy=\iint_D \operatorname{div}(u\nabla u)\,dx\,dy-\iint_D \operatorname{div}\|\nabla u\|^2dx\,dy\overset{\text{TD}}{=}\oint_{\partial D}u\,\frac{\partial u}{\partial\mathbf{n}}\,ds-\iint_D\|\nabla u\|^2dx\,dy\,,$$

$\downarrow$ $u\,\nabla u\cdot\mathbf{n}=u\,\frac{\partial u}{\partial\mathbf{n}}$.

c) Usando el teorema de la divergencia en el espacio: $\iiint_V u\,\Delta u\,dx\,dy\,dz=\iint_{\partial V}u\,\frac{\partial u}{\partial\mathbf{n}}\,dS-\iiint_V\|\nabla u\|^2dx\,dy\,dz$.

d) En una variable: $\int_a^b u\,u''dx=u\,u'\big]_a^b-\int_a^b(u')^2dx$ es una simple integración por partes.