Soluciones de problemas de Cálculo (grupo D - 15/16) — 4. Integrales de línea

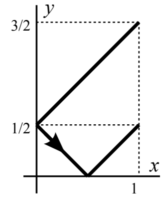
1. i) $y=f(x)$, $x\in[a,b]$ $\to$ $\mathbf{c}(x)=(x,f(x))$, $\mathbf{c}'(x)=(1,f'(x))$ $\to$ $L=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$.

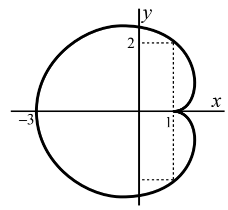
ii) $r=f(\theta)$, $\theta\in[\alpha,\beta]$ $\to$ $\mathbf{c}(\theta)=(f(\theta)\cos\theta,\, f(\theta)\,\mathrm{sen}\,\theta)$, $\mathbf{c}'(\theta)=(f'\cos\theta-f\,\mathrm{sen}\,\theta,\, f'\,\mathrm{sen}\,\theta+f\cos\theta)$

$\to$ $\|\mathbf{c}'(\theta)\|=\sqrt{f^2(\cos^2\theta+\mathrm{sen}^2\theta)+(f')^2(\cos^2\theta+\mathrm{sen}^2\theta)}$ $\to$ $L=\int_\alpha^\beta\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta$.

2. a) $(-t,\frac12-t)$, $t\in[-1,0]$; $(t,\frac12-t)$, $t\in[0,\frac12]$; $(t,t-\frac12)$, $t\in[\frac12,1]$

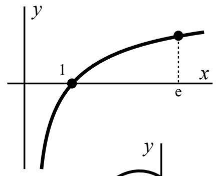
$\to$ $L=\int_{-1}^0\sqrt2\,dt+\int_0^{1/2}\sqrt2\,dt+\int_{1/2}^1\sqrt2\,dt=2\sqrt2$ (¡claro!).

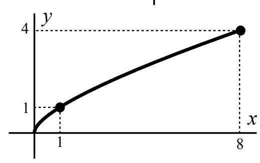




b) $\|2(\mathrm{sen}\,2t-\mathrm{sen}\,t,\cos t-\cos 2t)\|=2\sqrt2\sqrt{1-\cos t}=4|\mathrm{sen}\,\frac t2|$,

$L=8\int_0^\pi\mathrm{sen}\,\frac t2\,dt=-16\cos\frac t2\Big]_0^\pi=16$.

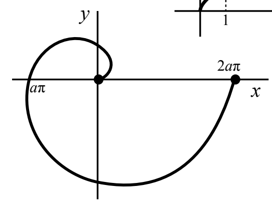




c) $L=\int_1^e\frac{\sqrt{x^2+1}}{x}\,dx \overset{u=x+\sqrt{x^2+1}}{=\cdots=} \sqrt{x^2+1}-\log\frac{1+\sqrt{x^2+1}}{x}\Big]_1^e\approx 2.0035$.

d) $L=\int_1^8\sqrt{1+\frac49x^{-2/3}}\,dx$ es difícil, pero parametrizando $x=y^{3/2}$,

$y\in[1,4]$, $L=\int_1^4\sqrt{1+\frac94y}\,dy=\frac{8}{27}[10^{3/2}-\frac18 13^{3/2}]\approx7.63$.



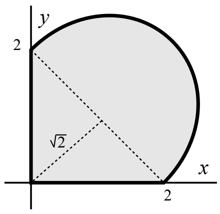
e) $L=\int_0^{2\pi}a\sqrt{\theta^2+1}\,d\theta=\frac a2[\theta\sqrt{\theta^2+1}+\log|\theta+\sqrt{\theta^2+1}|]_0^{2\pi}\approx21.3\,a$.

3. $r^2=2r\cos\theta+2r\,\mathrm{sen}\,\theta$, $x^2+y^2=2x+2y$, $(x-1)^2+(y-1)^2=2$, circunferencia.

Pasando de integrales: a) $A=\frac{\pi(\sqrt2)^2}{2}+\frac{2\cdot1}{2}=\pi+2$; b) $L=2+2+\frac{2\pi\sqrt2}{2}=4+\pi\sqrt2$.

Con integrales: a) $\int_0^{\pi/2}\int_0^{2(\cos\theta+\mathrm{sen}\,\theta)}r\,dr\,d\theta=\int_0^{\pi/2}(2+4\,\mathrm{sen}\,\theta\cos\theta)\,d\theta=\pi+2$.

b) $r^2+(r')^2=8$ $\to$ $L=2+2+\int_0^{\pi/2}\sqrt8\,d\theta=4+\pi\sqrt2$.

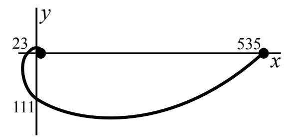


4. a) $f(x,y,z)=yz$, $\mathbf{c}(t)=(t,3t,2t)$, $\mathbf{c}'(t)=(1,3,2)$; $\int_{\mathbf c}f\,ds=\int_1^3 6t^2\sqrt{14}\,dt=2\sqrt{14}\,t^3\Big]_1^3=52\sqrt{14}$.

b) $f(x,y,z)=x+z$, $\mathbf c(t)=(t,t^2,\frac23t^3)$, $\mathbf c'(t)=(1,2t,2t^2)$; $\int_{\mathbf c} f\,ds=\int_0^1(t+\frac23t^3)(1+2t^2)\,dt=\frac12(t+\frac23t^3)^2\Big]_0^1=\frac{25}{18}$.

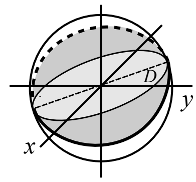
5. $L=\int_0^{2\pi}\sqrt{e^{2\theta}+e^{2\theta}}\,d\theta=\sqrt2\int_0^{2\pi}e^\theta d\theta=\sqrt2\,[e^{2\pi}-1]\approx756$.

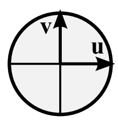
$T_{\text{media}}=\frac1L\int_0^{2\pi}e^\theta\sqrt{e^{2\theta}+e^{2\theta}}\,d\theta=\frac{\sqrt2}{L}\int_0^{2\pi}e^{2\theta}d\theta=\frac{\sqrt2}{2L}[e^{4\pi}-1]=\frac12[e^{2\pi}+1]\approx268$.



6. La intersección de $x^2+y^2+z^2=1$ y $x+y+z=0$ es una circunferencia C parametrizable con $\mathbf{c}(t)=\cos t\,\mathbf{u}+\mathrm{sen}\,t\,\mathbf{v}$, con $\mathbf{u}$, $\mathbf{v}$ vectores ortogonales unitarios contenidos en el plano.

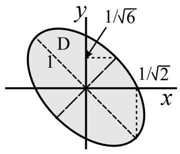
Por ejemplo, $\mathbf u=(\frac1{\sqrt2},-\frac1{\sqrt2},0)$, $\mathbf v=(\frac1{\sqrt6},\frac2{\sqrt6},-\frac1{\sqrt6})$. $\|\mathbf c'(t)\|=1$.





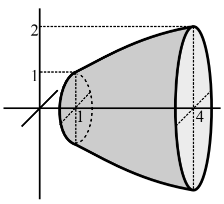
$M=\int_0^{2\pi}(\frac{\cos t}{\sqrt2}+\frac{\mathrm{sen}\,t}{\sqrt6})^2dt=\int_0^{2\pi}(\frac{\cos^2t}{2}+\frac{\mathrm{sen}\,t\cos t}{\sqrt3}+\frac{\mathrm{sen}^2t}{6})^2dt=\frac\pi2+\frac\pi6=\frac{2\pi}{3}$.

[La proyección de C sobre $z=0$ es $2x^2+2y^2+2xy=1$; de ella salen otras parametrizaciones con cálculos bastante más largos, por ejemplo: $(x,\frac12[-x\pm\sqrt{2-3x^2}\,],\frac12[x\pm\sqrt{2-3x^2}\,])$, o parametrizando distinto la elipse: $(\frac{\sqrt2}{\sqrt3}\,\mathrm{sen}\,t,\frac1{\sqrt2}\cos t-\frac1{\sqrt6}\,\mathrm{sen}\,t,(\frac1{\sqrt6}-\frac{\sqrt2}{\sqrt3})\,\mathrm{sen}\,t-\frac1{\sqrt2}\cos t)$].



7. a) (y^2,y), $y\in[1,2]$ ó $(x,x^{1/2})$, $x\in[1,4]$. Mejor con la primera:

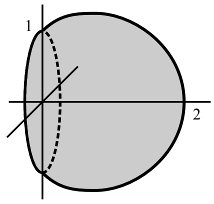
$A=2\pi\int_1^2 y\sqrt{1+4y^2}\,dy=\frac\pi6(1+4y^2)^{3/2}\Big]_1^2=\frac\pi6(17^{3/2}-5^{3/2})\approx30.8$.



b) $r=(1+\cos\theta)$, $\theta\in[0,\frac\pi2]$, $r'(\theta)=-\,\mathrm{sen}\,\theta$, $y=(1+\cos\theta)\,\mathrm{sen}\,\theta$,

$\|\mathbf c'(\theta)\|=\sqrt{r^2+(r')^2}=\sqrt{1+2\cos\theta+\cos^2\theta+\mathrm{sen}^2\theta}=\sqrt{2+2\cos\theta}$.

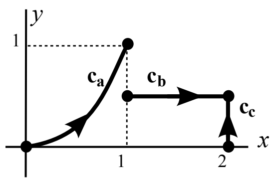
$A=2\sqrt2\,\pi\int_0^{\pi/2}(1+\cos\theta)^{3/2}\,\mathrm{sen}\,\theta\,d\theta=-\frac{4\sqrt2\,\pi}{5}(1+\cos\theta)^{5/2}\Big]_0^{\pi/2}=\frac{4\pi}{5}(8-\sqrt2)$ $\approx 16{,}55$.



8. a) $\mathbf{c}(x)=(x,x^2)$, $0\le x\le 1$, $\int_C (x^2+y^2)dx+dy=\int_0^1(x^2+x^4+2x)\,dx=\frac{23}{15}$.

b) $\mathbf{c}(x)=(x,\frac12)$, $1\le x\le 2$, $\int_C (x^2+y^2)dx+dy=\int_1^2(x^2+\frac14)\,dx=\frac{31}{12}$.

c) $\mathbf{c}(y)=(2,y)$, $0\le y\le\frac12$, $\int_C (x^2+y^2)dx+dy=\int_0^{1/2}dy=\frac12$.

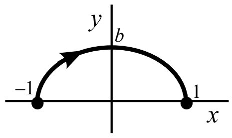


9. $\mathbf{c}(t)=(\cos t,b\,\mathrm{sen}\,t)$, $t\in[0,\pi]$, recorre $b^2x^2+y^2=b^2$ en sentido opuesto.

$T(b)=-\int_0^{\pi}(3b^2\,\mathrm{sen}^2t+2,16\cos t)\cdot(-\,\mathrm{sen}\,t,b\cos t)\,dt=4b^2-8b\pi+4$. $T'(b)=8(b-\pi)$

$||$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$3b^2\,\mathrm{sen}\,t(1-\cos^2t)-8b(1+\cos 2t)+2\,\mathrm{sen}\,t$ $\qquad$ T mínimo si $b=\pi$.



10. $\mathbf{f}(x,y)=(xy,0)$ entre $(-1,0)$ y $(1,0)$. Usamos en todas el parámetro x, $x\in[-1,1]$:

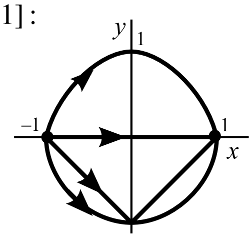
a) $\mathbf{c}(x)=(x,0)$, $\mathbf{c}'=(1,0)$. $\int_{-1}^1 0\,dx=0$.

b) $\mathbf{c}(x)=(x,1-x^2)$, $\mathbf{c}'=(1,-2x)$. $\int_{-1}^1(x-x^3)\,dx=0$.

c) $\mathbf{c}(x)=(x,|x|-1)$, $\mathbf{c}'=\begin{cases}(1,-1), & x<0\\(1,1), & x>0\end{cases}$. $\int_{-1}^0(-x^2-x)\,dx+\int_0^1(x^2-x)\,dx=0$.

d) $\mathbf{c}(x)=(x,-\sqrt{1-x^2})$, $\mathbf{c}'=(1,x(1-x^2)^{-1/2})$. $\int_{-1}^1-x\sqrt{1-x^2}\,dx=0$.

Pero no es gradiente de nigún campo escalar pues $f_y=x\not\equiv0=g_x$ (sobre otros caminos no se anula).



11. a) $\mathbf{c}(t)=(t^3,2t^2)$. $\mathbf{c}'(t)=(3t^2,4t)$, $\|\mathbf{c}'(t)\|=\sqrt{9t^4+16t^2}=|t|\sqrt{9t^2+16}$.

Longitud $L=\int_{-1}^0(-t)\sqrt{9t^2+16}\,dt=\frac{1}{27}\Big[-(9t^2+16)^{3/2}\Big]_{-1}^0=\frac{5^3-4^3}{27}=\frac{61}{27}$.

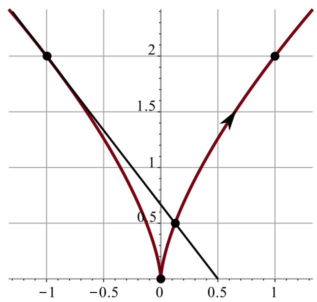
b) Tangente en $(-1,2)$: $\mathbf{c}(-1)=(-1,2)$, $\mathbf{c}'(-1)=(3,-4)$ $\to$ $\mathbf{x}(s)=(3s-1,2-4s)$.

Corta la curva para t y s que cumplan: $t^3=3s-1$, $2t^2=2-4s$ $\to$

$(3s-1)^2=(1-2s)^3$, $8s^3-3s^2=0$, $s=\frac38\to(\frac12,\frac18)$ [y $s=0\to(-1,2)$].

$\Big[$O bien, la curva $y=2x^{2/3}$ y la recta $y=\frac{2-4x}{3}$ se cortan si $27x^2\overset{\uparrow}{=}(1-2x)^3\Big]$.

c) $h(x,y)=e^{2x+y}$, $\int_{\overline{c}}\nabla h\cdot d\overline{s}=h(1,2)-h(0,0)=e^4-1=\int_0^1(6t^2+4t)\,e^{2t^3+2t^2}dt=e^{2t^3+2t^2}\Big]_0^1$ (cálculo innecesario).

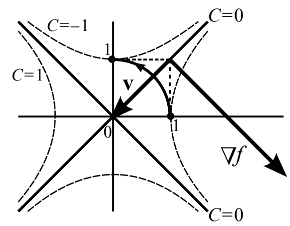


12. **a]** $f(x,y)=x^2-y^2=0$ $\to$ las rectas $y=\pm x$. [El resto, hipérbolas]. $\nabla f(x,y)=(2x,-2y)$, $\nabla f(1,1)=(2,-2)$. $D_{\mathbf{u}}f(1,1)=(2,-2)\cdot(-1,-1)=0$. $\Delta f(x,y)=f_{xx}+f_{yy}=2-2=0$.

b] Como $\mathbf{g}=(2x,-2y)=\nabla f$, será $\int_{\mathbf{c}}\mathbf{g}\cdot d\mathbf{s}=f(0,1)-f(1,0)=-2$. Directamente:

$\mathbf{c}(t)=(\cos t,\mathrm{sen}\,t)$, $t\in[0,\frac{\pi}{2}]$, $\int_{\mathbf{c}}\mathbf{g}\cdot d\mathbf{s}=\int_0^{\pi/2}(2c,-2s)\cdot(-s,c)\,dt=-2\,\mathrm{sen}^2t\big]_0^{\pi/2}=-2$.

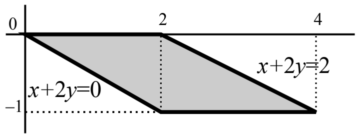
O bien: $\mathbf{c}(t)=(t,\sqrt{1-t^2})$, $t\in[1,0]$, $\int_{\mathbf{c}}\mathbf{g}\cdot d\mathbf{s}=\int_1^0(2t,-2\sqrt{\cdot})\cdot(1,\frac{-t}{\sqrt{\cdot}})\,dt=\int_1^0 4t\,dt=-2$.



13. D limitado por $y=-2$, $y=0$, $x+2y=0$ y $x+2y=2$.

a) $\iint_D(x+2y)\,dx\,dy=\int_{-1}^0\int_{-2y}^{2-2y}(x+2y)\,dx\,dy=\int_{-1}^0\Big([\frac{x^2}{2}]_{-2y}^{2-2y}+4y\Big)dy=\int_{-1}^0 2\,dy=2$.

O bien: $\begin{matrix}u=x+2y,\ x=u-2v\\ v=y\end{matrix}$, $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1&-2\\0&1\end{vmatrix}=1$ $\to$ $\int_0^2\int_{-1}^0u\,dv\,du=\int_0^2u\,du=2$.



b) Como $\mathbf{f}(x,y)=(1,\cos y)$ cumple $(1)_y=0=(\cos y)_x$ $\Rightarrow$ deriva de potencial ($U=x+\mathrm{sen}\,y$) $\Rightarrow\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}=0$.

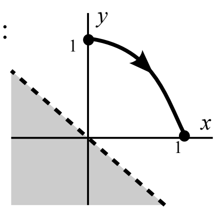
Directamente (largo): $\mathbf{c_1}=(t,-\frac t2)$, $t\in[0,2]$, $\mathbf{c_2}=(t,-1)$, $t\in[2,4]$, $\mathbf{c_3}=(t,1-\frac t2)$, $t\in[4,2]$, $\mathbf{c_4}=(t,0)$, $t\in[2,0]$,

$\int_0^2[1-\frac12\cos\frac t2]\,dt+\int_2^4dt+\int_4^2[1-\frac12\cos(1-\frac t2)]\,dt+\int_2^0dt=-\frac14\,\mathrm{sen}\,1+\frac14\,\mathrm{sen}\,1=0$.

14. Para $\mathbf{f}(x,y)=(-\frac{y}{(y+x)^2},\frac{x}{(y+x)^2})$ es $f_y=\frac{y-x}{(y+x)^3}=g_x$, y hallamos un potencial en $y+x>0$:

$U=\int\frac{-y\,dx}{(y+x)^2}=\frac{y}{y+x}+p(y)$
$U=\int\frac{x\,dy}{(y+x)^2}=\frac{y}{y+x}-1+q(x)$ $\Rightarrow U=\frac{y}{y+x}\Rightarrow\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=U(1,0)-U(0,1)=0-1=-1$.

Directamente: $\mathbf{c}=(1-t^2,t)$, $t\in[1,0]$, $\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=\int_1^0\frac{1+t^2}{(1+t-t^2)^2}\,dt=\cdots=\frac{t}{1+t-t^2}\Big]_1^0=-1$.



15. $\mathbf{f}(x,y)=(5-\frac{x}{x^2+y^2})\,\mathbf{i}-\frac{y}{x^2+y^2}\,\mathbf{j}$ es C^1 en $\mathbf{R}^2-\{(0,0)\}$, con agujero, pero existe potencial $U=5x-\frac12\log(x^2+y^2)$.

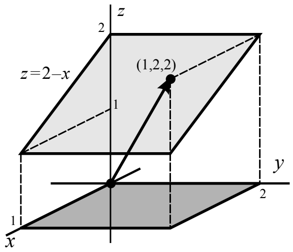
$\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=U(1,1)-U(-1,1)=10$. Directamente: $\int_{-\pi/4}^{\pi/4}(5-\frac{\sqrt2\cos t}{2},-\frac{\sqrt2\,\mathrm{sen}\,t}{2})\cdot\sqrt2\,(-\,\mathrm{sen}\,t,\cos t)\,dt=10$.

16. $\mathbf{F}(x,y,z)=(x,y,z)$. a) $\mathbf{c}(t)=(t,t,t)$, $t\in[0,1]$; $\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}=\int_0^1(t,t,t)\cdot(1,1,1)\,dt=\int_0^1 3t\,dt=\frac{3}{2}$.

b) $\mathbf{c}(t)=(\cos t,\operatorname{sen} t,0)$, $t\in[0,2\pi]$; $\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}=\int_0^{2\pi}(\cos t,\operatorname{sen} t,0)\cdot(-\operatorname{sen} t,\cos t,0)\,dt=\int_0^{2\pi}0\,dt=0$.

17. $g(x,y,z)=y\,e^{2x-z}$. a) En $[0,1]\times[0,2]$ es $z=2-x>z=0$ (no se precisa el dibujo).

$\iiint_V g=\int_0^2\int_0^1\int_0^{2-x}y\,e^{2x-z}\,dz\,dx\,dy=2\int_0^1[e^{2x}-e^{3x-2}]\,dx=[e^{2x}-\frac{2}{3}e^{3x-2}]_0^1=\frac{3e^2-3+2e+2e^{-2}}{3}$.

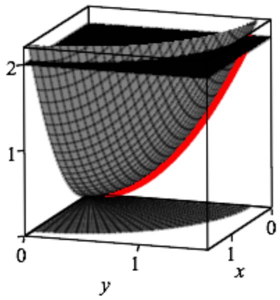


b) i) $\mathbf{c}(t)=(t,2t,2t)$, $\mathbf{c}'=(1,2,2)$, $\|\mathbf{c}'\|=3$. $\int_{\mathbf{c}}g\cdot ds=\int_0^1 g(\mathbf{c}(t))\|\mathbf{c}'(t)\|\,dt=\int_0^1 6t\,dt=3$.

ii) Más fácil: $\int_{\mathbf{c}}\nabla g\cdot d\mathbf{s}=g(1,2,2)-g(0,0,0)=2$.

Directamente: $\nabla g=(2y\,e^{2x-z},e^{2x-z},-ye^{2x-z})$, $\int_0^1(4t,1,-2t)\cdot(1,2,2)\,dt=\int_0^1 2\,dt=2$.

18. a) Paraboloide y plano se cortan en la circunferencia de radio $\sqrt{2}$. Cilíndricas:



$\iiint_V f=\int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{r^2}^2 r^2\operatorname{sen}\theta\,dz\,dr\,d\theta=[-\cos\theta]_0^{\pi/2}\int_0^{\sqrt{2}}(2r^2-r^4)\,dr=\left[\frac{2r^2}{3}-\frac{r^5}{5}\right]_0^{\sqrt{2}}=\frac{8}{15}\sqrt{2}$.

$\iiint_V f=\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{x^2+y^2}^2 y\,dz\,dy\,dx=\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}(2y-x^2y-y^3)\,dy\,dx$

$=\int_0^{\sqrt{2}}[\frac{1}{2}(2-x^2)^2-\frac{1}{4}(2-x^2)^2]\,dx=\int_0^{\sqrt{2}}(1-x^2+\frac{1}{4}x^4)\,dx=\sqrt{2}\,(1-\frac{2}{3}+\frac{1}{5})=\frac{8}{15}\sqrt{2}$.

b) Parametrizaciones de C : $\mathbf{c}(t)=(0,t,t^2)$, $t\in[0,\sqrt{2}]$ o $\mathbf{c}_*(t)=(0,\sqrt{t},t)$, $t\in[0,2]$.

i) $\mathbf{c}'(t)=(0,1,2t)$, $\|\mathbf{c}'\|=\sqrt{1+4t^2}$, $\int_C f\,ds=\int_0^{\sqrt{2}}t\,(1+4t^2)^{1/2}dt=\frac{1}{12}(1+4t^2)^{3/2}]_0^{\sqrt{2}}=\frac{13}{6}$.

$\mathbf{c}_*'(t)=(0,\frac{1}{2\sqrt{t}},1)$, $\|\mathbf{c}'\|=\sqrt{\frac{1}{4t}+1}$, $\int_C f\,ds=\int_0^2\frac{1}{2}(1+4t)^{1/2}dt=\frac{1}{12}(1+4t)^{3/2}]_0^2=\frac{13}{6}$.

ii) $\int_C\nabla f\cdot d\bar{s}=f(0,\sqrt{2},2)-f(0,0,0)=\sqrt{2}$. $\left[\text{O bien: } \int_0^{\sqrt{2}}(0,1,0)\cdot(0,1,2t)\,dt=\int_0^2(0,1,0)\cdot(0,\frac{1}{2\sqrt{t}},1)\,dt=\sqrt{2}\right]$.

19. $\operatorname{div}\bar{f}=y+0-y=\boxed{0}$. $\operatorname{rot}\bar{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ xy&2x&-yz\end{vmatrix}=-z\,\mathbf{i}+0\,\mathbf{j}+(1-y)\,\mathbf{k}=\boxed{(-z,0,2-y)}$ [no deriva de un potencial].

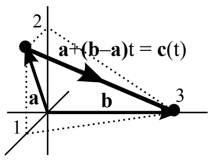
$\int_{\bar{c}}\bar{f}\cdot d\bar{s}=\int_0^\pi(cs,2c,-s)\cdot(-s,c,0)\,dt=\int_0^\pi(2\cos^2 t-\operatorname{sen}^2 t\cos t)\,dt=\pi+\left[\frac{1}{2}\operatorname{sen}2t-\frac{1}{3}\operatorname{sen}^3 t\right]_0^\pi=\boxed{\pi}$.

Como $\|\bar{c}'\|=\sqrt{\operatorname{sen}^2 t+\cos^2 t+0}=1$, la longitud de la curva es $L=\int_0^\pi 1\,dt=\boxed{\pi}$.

20. **a]** $\mathbf{c}(t)=(1,1-t,3t)$, $t\in[0,1]\to\int_{\mathbf{c}}\bar{f}\cdot d\bar{s}=\int_0^1(e^{-3t},1,-e^{-3t})\cdot(0,-1,3)\,dt=\int_0^1(-1-3e^{-3t})\,dt=\boxed{e^{-3}-2}$.

b] $\operatorname{rot}\bar{f}=\bar{0}$, $\bar{f}\in C^1$ $\Rightarrow$ hay potencial. $\begin{matrix}U=xe^{-z}+p(y,z)\\ U=y+q(x,z)\\ U=xe^{-z}+r(x,y)\end{matrix}$, $\boxed{U=xe^{-z}+y}$ $[\int_{\mathbf{c}}\bar{f}\cdot d\bar{s}=U(1,0,3)-U(1,1,0)=e^{-3}-2]$.

21. $\mathbf{F}(x,y,z)=(1,2yz,y^2)$, $\mathbf{c}(t)=(1,0,2)+t(-1,3,-2)=(1-t,3t,2-2t)$, $t\in[0,1]$.



$\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}=\int_0^1(1,12t-12t^2,9t^2)\cdot(-1,3,-2)\,dt=\int_0^1(-1+36t-54t^2)\,dt=-1$.

Como $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ 1&2yz&y^2\end{vmatrix}=(2y-2y)\,\mathbf{i}+0\,\mathbf{j}+0\,\mathbf{k}=\mathbf{0}$ y $\mathbf{F}\in C^1(\mathbf{R}^3)$ existirá un potencial.

[Y la integral será -1 para toda curva].

$\begin{matrix}U=x+p(y,z)\\ U=y^2z+q(x,z)\\ U=y^2z+r(x,y)\end{matrix}$ $\Rightarrow$ $U=x+y^2z$ $\Rightarrow\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}=U(0,3,0)-U(1,0,2)=0-1=-1$ (de otra forma).

22. $\mathbf{f}(x,y,z)=(z^2,2y,cxz)$. a) $\operatorname{div}\mathbf{f}=2+cx$. $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ z^2&2y&cxz\end{vmatrix}=(0,2z-cz,0)$.

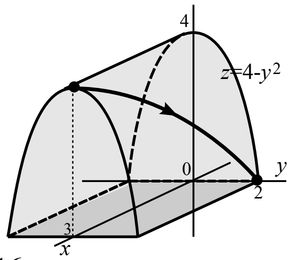
b) Para $c=2$ es $\operatorname{rot}\mathbf{f}=\mathbf{0}$. Como $\mathbf{f}\in C^1(\mathbf{R}^3)$, existe el potencial: $\begin{matrix}U_x=z^2\to U=xz^2+p(y,z)\\ U_y=2y\to U=y^2+q(x,z)\\ U_z=2xz\to U=xz^2+r(x,y)\end{matrix}$, $U=xz^2+y^2$.

c] Por tanto, $\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=U(1,0,1)-U(0,0,0)=1$, sin necesidad de hacer ninguna integral de línea.

[Una parametrización sería: $\mathbf{c}(t)=(t,0,t)$, $t\in[0,1]\to\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=\int_0^1(t^2,0,2t^2)\cdot(1,0,1)\,dt=\int_0^1 3t^2\,dt=1$].

23. a) $\operatorname{div}\mathbf{f}=2x-2$. $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ y^2&xy&-2z\end{vmatrix}=0\,\mathbf{i}+0\,\mathbf{j}+(2y-2y)\,\mathbf{k}=(0,0,0)\overset{\mathbf{f}\in C^1}{\Longrightarrow}$ $\mathbf{f}$ deriva de un potencial.

$\nabla(\operatorname{div}\mathbf{f})=(2,0,0)$. $\Delta(\mathbf{f}\cdot\mathbf{f})=\Delta(y^4+4x^2y^2+4z^2)=20y^2+8x^2+8$.



b) $z=4-y^2$ corta $z=0$ en las rectas $y=\pm 2$. V es el del dibujo. Por tanto:

$\iiint_V \operatorname{div}\mathbf{f}=\int_0^3\int_{-2}^2\int_0^{4-y^2}(2x-2)\,dz\,dy\,dx=\int_0^3(2x-2)\,dx\int_{-2}^2(4-y^2)\,dy=32$.

c) Por ser conservativo, podemos hallar la integral hallando el potencial U:

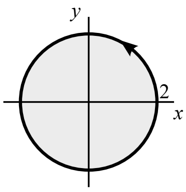
$U_x=y^2\ \to\ U=xy^2+p(y,z)$
$U_y=2xy\ \to\ U=xy^2+q(x,z)$, $\quad U=xy^2-z^2\ \Rightarrow\ \int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=U(0,2,0)-U(3,0,4)=16$.
$U_z=-2z\ \to\ U=-z^2+r(x,y)$

O directamente: $\int_{\mathbf{c}^*}\mathbf{f}\cdot d\mathbf{s}=\int_0^2(t^2,6t-3t^2,2t^2-8)\cdot(-\frac{3}{2},1,-2t)\,dt=\int_0^2(22t-\frac{9}{2}t^2-4t^3)\,dt=11t^2-\frac{3}{2}t^3-t^4\Big]_0^2=16$.

O hallar la integral siguiendo un camino más sencillo, por ejemplo el segmento que une los puntos:

$\mathbf{c}^*(t)=(3-3t,2t,4-4t)$, $t\in[0,1]$ $\to$ $\int_0^1(4t^2,12t-12t^2,8t-8)\cdot(-3,2,-4)\,dt=\int_0^1(32-8t-36t^2)\,dt=16$.

24. $\mathbf{g}(x,y)=(1,xy^2)$. $g_x-f_y=y^2$. No deriva de un potencial. i) $\mathbf{c}(t)=(2\cos t,2\operatorname{sen}t)$, $t\in[0,2\pi]$.

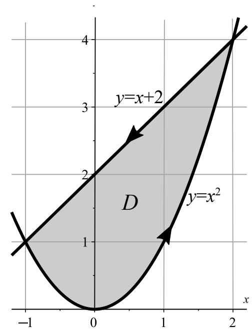


$\to\int_{\mathbf{c}}\mathbf{g}\cdot d\mathbf{s}=\int_0^{2\pi}(1,8cs^2)\cdot(-2s,2c)\,dt=2\cos t\Big]_0^{2\pi}+\int_0^{2\pi}4\operatorname{sen}^2 2t\,dt=\int_0^{2\pi}(2-2\cos 4t)\,dt=4\pi$.

ii) Green: $\iint_\circ y^2\,dx\,dy=\int_0^{2\pi}\int_0^2 r^3\operatorname{sen}^2\theta\,dr\,d\theta=[\frac{1}{4}r^4]_0^2\cdot\frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta=4\pi$.

$\Big[$En cartesianas mucho más largo: $\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}y^2\,dy\,dx=\frac{2}{3}\int_{-2}^2(4-x^2)^{3/2}dx=\cdots\Big]$.

25. $\mathbf{f}(x,y)=(-xy,y)$. $g_x-f_y=x$. $\iint_D x=\int_{-1}^2\int_{x^2}^{2+x}x\,dy\,dx=\int_{-1}^2(2x+x^2-x^3)\,dx=\frac{9}{4}$.



O bien: $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}x\,dx\,dy+\int_1^4\int_{y-2}^{\sqrt{y}}x\,dx\,dy=0+\frac{1}{2}\int_1^4(5y-y^2-4)\,dy=\frac{9}{4}$.

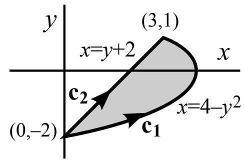
Parametrizamos ∂D: $\mathbf{c}_1(x)=(x,x^2)$, $x\in[-1,2]$. $\mathbf{c}_2(x)=(x,x+2)$, $x\in[2,-1]$

[o $\mathbf{c}_2(y)=(y-2,y)$, $y\in[4,1]$, o $\mathbf{c}_2(t)=(2-3t,4-3t)$, $t\in[0,1]$].

$\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{-1}^2(-x^3,x^2)\cdot(1,2x)\,dx+\int_2^{-1}(-x^2-2x,2+x)\cdot(1,1)\,dx$

$=\int_{-1}^2x^3\,dx+\int_{-1}^2(x^2+x-2)\,dx=\frac{15}{4}+\frac{9}{3}+\frac{3}{2}-6=\frac{15}{4}-\frac{3}{2}=\frac{9}{4}$.

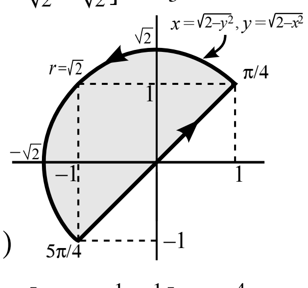
26. a) $\mathbf{f}(x,y)=(y^2,2x)$, $\mathbf{c_1}=(4-t^2,t)$, $t\in[-2,1]$, $\mathbf{c_2}=(t,t-2)$, $t\in[0,3]$ (sentido opuesto).



$\int_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{-2}^1(-2t^3+8-2t^2)\,dt-\int_0^3(t^2-2t+4)\,dt=26-\frac{1}{2}-12=\frac{27}{2}$.

$g_x-f_y=2-2y$, $\int_{-2}^1\int_{y+2}^{4-y^2}(2-2y)\,dx\,dy=\int_{-2}^1(4-6y+2y^3)\,dy=\frac{27}{2}$.

b) $\mathbf{f}(x,y)=(y^2,xy)$. $\iint_D -y\,dx\,dy=-\int_{\pi/4}^{5\pi/4}\int_0^{\sqrt{2}}r^2\operatorname{sen}\theta=[\frac{1}{3}r^3]_0^{\sqrt{2}}\,[\cos\theta]_{\pi/4}^{5\pi/4}=\frac{2\sqrt{2}}{3}\Big[-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\Big]=-\frac{4}{3}$.



En cartesianas (de las dos formas) es más complicado. Por ejemplo:

$\iint_D f=-\int_{-\sqrt{2}}^{-1}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}y\,dy\,dx-\int_{-1}^1\int_{-x}^{\sqrt{2-x^2}}y\,dy\,dx=0+\int_{-1}^1[x^2-1]\,dx=-\frac{4}{3}$.

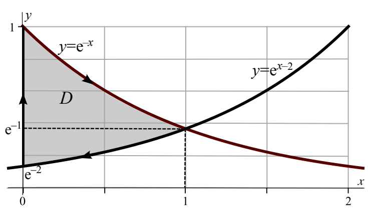
Parametrizaciones sencillas: $\mathbf{c_1}(t)=(t,t)$, $t\in[-1,1]$ (en sentido correcto).

$\mathbf{c_2}(t)=(\sqrt{2}\cos t,\sqrt{2}\operatorname{sen}t)$, $t\in[\frac{\pi}{4},\frac{5\pi}{4}]$ (también en buen sentido).

$\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{-1}^1(t^2,t^2)\cdot(1,1)\,dt+\int_{\pi/4}^{5\pi/4}(2\operatorname{sen}^2t,2\operatorname{sen}t\cos t)\cdot(-\sqrt{2}\operatorname{sen}t,\sqrt{2}\cos t)$

$=\int_{-1}^1 2t^2\,dt+2\sqrt{2}\int_{\pi/4}^{5\pi/4}[sc^2-s^3]\,dt=\frac{4}{3}+2\sqrt{2}\,[\cos t-\frac{2}{3}\cos^3t]_{\pi/4}^{5\pi/4}=\frac{4}{3}+2[1+1-\frac{1}{3}-\frac{1}{3}]=-\frac{4}{3}$.

27. a) $\int_0^1\int_{e^{x-2}}^{e^{-x}}x\,e^x\,dy\,dx=\int_0^1[x-xe^{2x-2}]\,dx=\frac{x^2}{2}-\frac{x}{2}e^{2x-2}\Big]_0^1+\frac{1}{2}\int_0^1e^{2x-2}dx=\frac{1-e^{-2}}{4}$.



b) $g_x-f_y=-x\,e^x$. Según Green, la $\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}$ vale lo de arriba. Directamente:

$\mathbf{c}_1(t)=(0,t)$, $y\in[e^{-2},1]$; $\mathbf{c}_2(t)=(t,e^{-t})$, $t\in[0,1]$; $\mathbf{c}_3(t)=(t,e^{t-2})$, $t\in[1,0]$.

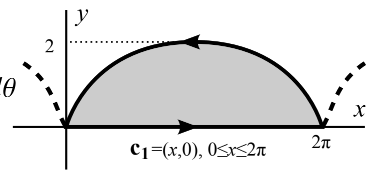
$\int_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{e^{-2}}^1(0,1)\cdot(0,1)\,dt+\int_0^1(t,1)\cdot(1,-e^{-t})\,dt+\int_1^0(te^{2t-2},1)\cdot(1,e^{t-2})\,dt$

$=1-e^{-2}+\int_0^1(t-e^{-t})\,dt+\int_1^0(te^{2t-2}+e^{t-2})\,dt=\frac{3}{2}-e^{-2}+[e^{-t}]_0^1+[\frac{t}{2}e^{2t-2}-\frac{1}{4}e^{2t-2}+e^{t-2}]_1^0=\frac{1-e^{-2}}{4}$.

28. $x=\theta-\operatorname{sen}\theta\,,\ y=1-\cos\theta\,,\ 0\le\theta\le 2\pi$. Utilizando Green:

$$A=\frac{1}{2}\oint_{\partial D} x\,dy-y\,dx=\frac{1}{2}\int_0^{2\pi}0-\frac{1}{2}\int_0^{2\pi}\left[(\theta-\operatorname{sen}\theta)\operatorname{sen}\theta-(1-\cos\theta)^2\right]d\theta$$

$$=\int_0^{2\pi}\left(1-\cos\theta-\tfrac{1}{2}\theta\operatorname{sen}\theta\right)d\theta=3\pi\,.$$



29. $\mathbf{g}(x,y)=(x^2,-2xy)$. **Green**. $\iint_D (g_x-f_y)=\int_0^2\int_0^{4-2x}(-2y)\,dy\,dx=-\int_0^2(4-2x)^2\,dx=\frac{1}{6}(4-2x)^3\Big]_0^2=-\frac{32}{3}$.

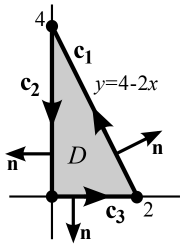
La ∂D está formada por 3 segmentos fáciles de parametrizar. Por ejemplo:

$\mathbf{c_1}(t)=(t,4-2t)\,,\ t\in[2,0]$ [para ir en sentido antihorario].

$\mathbf{c_2}(t)=(0,t)\,,\ t\in[4,0]$. $\mathbf{c_3}(t)=(t,0)\,,\ t\in[0,2]$. Entonces:

$$\oint_{\partial D}\mathbf{g}\cdot d\mathbf{s}=\int_{\mathbf{c_1}}+\int_{\mathbf{c_2}}+\int_{\mathbf{c_3}}=\int_4^0(t^2,4t^2-8t)\cdot(1,-2)\,dt+\int_4^0 0\,dt+\int_0^2(t^2,0)\cdot(1,0)\,dt$$

$$=\int_2^0(16t-7t^2)\,dt+0+\int_0^2 t^2\,dt=-32+\frac{56}{3}+\frac{8}{3}=-\frac{32}{3}\,.$$

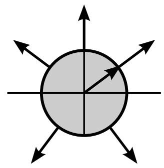


Divergencia. $\iint_D \operatorname{div}\mathbf{f}\,dx\,dy=\iint_D(2x-2x)\,dx\,dy=0$. Las normales unitarias exteriores $\mathbf{n}$ a cada segmento son, respectivamente: $\left(\frac{2}{\sqrt5},\frac{1}{\sqrt5}\right)$, $(-1,0)$ y $(0,-1)$. Sus normas $\|\mathbf{c}_\mathbf{k}'(t)\|$ son: $\sqrt5$, 1 y 1 . Por tanto:

$$\oint_{\partial D}\mathbf{g}\cdot\mathbf{n}\,ds=\int_0^4(t^2,4t^2-8t)\cdot(2,1)\,dt+\int_0^4 0\,dt+\int_0^2(t^2,0)\cdot(0,-1)\,dt=\int_0^2(6t^2-8t)\,dt-\int_0^2 0\,dt=16-16=0\,.$$

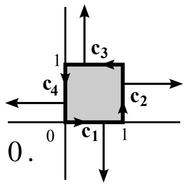
30. i) $\mathbf{f}(x,y)=(x,y)\,,\ D=\{x^2+y^2\le 1\}$. $\mathbf{f}\cdot\mathbf{n}=1$ (dos vectores unitarios en el mismo sentido).

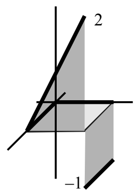
$$\oint_{\partial D}1\,ds\underset{\substack{\uparrow\\ \text{longitud}}}{=}2\pi=\iint_D 2\,dx\,dy\underset{\substack{\uparrow\\ \text{2 veces el área}}}{=}2\pi\,.$$



ii) $\mathbf{f}(x,y)=(2xy,-y^2)$, $\operatorname{div}\mathbf{f}=0$, $\iint_D \operatorname{div}\mathbf{f}=0$.

$$\oint_{\partial D}\mathbf{f}\cdot\mathbf{n}\,ds=\int_{\mathbf{c_1}}y^2ds+\int_{\mathbf{c_2}}2xy\,ds+\int_{\mathbf{c_3}}y^2ds+\int_{\mathbf{c_4}}2xy\,ds=0+\int_0^1 2t\,dt-\int_0^1 1\,dt-0=0\,.$$





[Hasta aquí el control 2].