Soluciones de problemas de Cálculo (grupo D - 15/16) — 3. Integrales múltiples

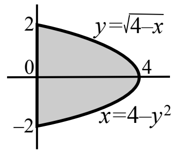
1. a) $\int_0^1\int_0^1 (x^2+y^2)\,dx\,dy = \int_0^1\int_0^1 x^2dx\,dy + \int_0^1\int_0^1 y^2dy\,dx = \int_0^1 x^2dx + \int_0^1 y^2dy = \frac{2}{3}$.

b) $\int_0^1\int_0^1 y\,e^{xy}dx\,dy = \int_0^1 [e^{xy}]_0^1\,dy = \int_0^1 [e^y-1]\,dy = e-1-1 = e-2$.

c) $\int_0^1\int_0^1 (xy)^2\cos x^3\,dx\,dy = \int_0^1 y^2[\frac{1}{3}\operatorname{sen} x^3]_0^1\,dy = \frac{1}{3}\operatorname{sen} 1\int_0^1 y^2dy = \frac{1}{9}\operatorname{sen} 1$.

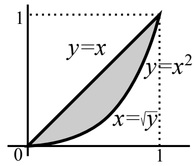
2. a) $\int_1^2\int_1^2 \log(xy)\,dx\,dy = \int_1^2 \log x\,dx + \int_1^2 \log y\,dy = 2[2\log 2-1]$, pues $\int_1^2 \log s\,ds = s\log s]_1^2 - \int_1^2 1\,ds = 2\log 2-1$.

b) $\int_{-2}^2\int_0^{4-y^2} x^3y\,dx\,dy = \frac{1}{4}\int_{-2}^2 \underbrace{y\,(4-y^2)^4}_{\text{impar}}\,dy = 0$. O bien $\int_0^4\int_{-\sqrt{4-x}}^{\sqrt{4-x}} x^3y\,dy\,dx = \int_0^4 0\,dx = 0$.



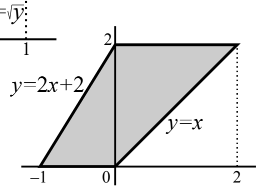
c) $\int_0^1\int_{x^2}^x xy\,dy\,dx = \int_0^1 x\,[\frac{y^2}{2}]_{x^2}^x\,dx = \frac{1}{2}\int_0^1 [x^3-x^5]\,dx = \frac{1}{2}[\frac{1}{4}-\frac{1}{6}] = \frac{1}{24}$.

O bien $\int_0^1\int_y^{\sqrt{y}} xy\,dx\,dy = \int_0^1 y\,[\frac{x^2}{2}]_y^{\sqrt{y}}\,dy = \frac{1}{2}\int_0^1 [y^2-y^3]\,dy = \frac{1}{2}[\frac{1}{3}-\frac{1}{4}] = \frac{1}{24}$.



d) $\int_0^2\int_{y/2-1}^y e^{x-y}\,dx\,dy = \int_0^2 [1-e^{-1-y/2}]\,dy = 2[1-\frac{1}{e}+\frac{1}{e^2}]$, o más largo:

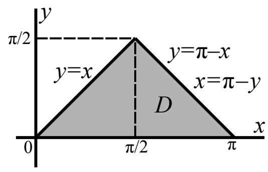
$\int_{-1}^0\int_0^{2x+2} e^{x-y}\,dy\,dx + \int_0^2\int_x^2 e^{x-y}\,dy\,dx = \int_{-1}^0 [e^x - e^{-x-2}]\,dx + \int_0^2 [1-e^{x-2}]\,dx$



e) Mejor: $\int_0^{\pi/2}\int_y^{\pi-y} \operatorname{sen} x\,dx\,dy = \int_0^{\pi/2} [\cos y - \underbrace{\cos(\pi-y)}_{=-\cos y}]\,dy = \big[2\operatorname{sen} y\big]_0^{\pi/2} = 2$.

Peor: $\int_0^{\pi/2}\int_0^x \operatorname{sen} x\,dy\,dx + \int_{\pi/2}^{\pi}\int_0^{\pi-x} \operatorname{sen} x\,dy\,dx = \int_0^{\pi/2} x\operatorname{sen} x\,dx + \int_{\pi/2}^{\pi} (\pi-x)\operatorname{sen} x\,dx$

$= -[x\cos x]_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx - [(\pi-x)\cos x]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\cos x\,dx = 2$.



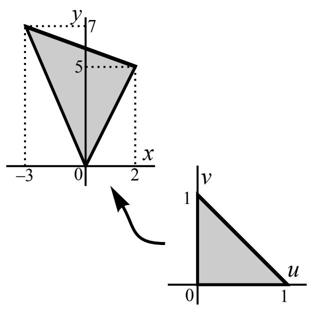
f) $\int_{-3}^0\int_{-7x/3}^{(29-2x)/5} x\,dy\,dx + \int_0^2\int_{5x/2}^{(29-2x)/5} x\,dy\,dx = \int_{-3}^0 [\frac{29x^2}{15}+\frac{29x}{5}] + \int_0^2 [\frac{29x}{5}-\frac{29x^2}{10}]$

$= \frac{29}{5}[\frac{x^3}{9}+\frac{x^2}{2}]_{-3}^0 + \frac{29}{5}[\frac{x^2}{2}-\frac{x^3}{6}]_0^2 = \frac{29}{5}[3-\frac{9}{2}+2-\frac{4}{3}] = -\frac{29}{6}$.

Con un cambio lineal podemos llevar el triángulo a uno más sencillo; por ejemplo el que lleva $(1,0)$ a $(2,5)$ y $(0,1)$ a $(-3,7)$, es decir, el dado por la matriz:

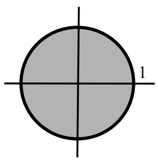
$\mathbf{A} = \begin{pmatrix}2 & -3\\ 5 & 7\end{pmatrix}$, o sea, $\begin{pmatrix}x\\ y\end{pmatrix} = \mathbf{A}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}2u-3v\\ 5u+7v\end{pmatrix}$; $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix}2 & -3\\ 5 & 7\end{vmatrix} = 29$;

$\int_0^1\int_0^{1-u} 29(2u-3v)\,dv\,du = 29\int_0^1 \Big[2u(1-u)-\frac{3(1-u)^2}{2}\Big]du = -\frac{29}{6}$.



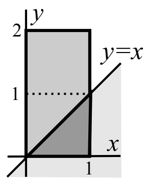
g) $\int_{-1}^1 x^3\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 2\int_{-1}^1 x^3\sqrt{1-x^2}\,dx = 0$ [integral de función impar en recinto simétrico].

$\int_0^{2\pi}\cos^3\theta\int_0^1 r^4\,dr\,d\theta = \frac{1}{5}\int_0^{2\pi}\cos^3\theta\,d\theta = 0$



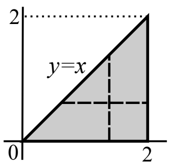
3. Como el integrando tiene dos expresiones en R, debemos hallar dos integrales:

$\int_0^1\int_0^x (x-y)\,dy\,dx + \int_0^1\int_x^2 (y-x)\,dy\,dx = \int_0^1 \frac{x^2}{2}\,dx + \int_0^1 [2-2x+\frac{x^2}{2}]\,dx = \frac{4}{3}$.



4. $\int_0^2\int_y^2 e^{x^2}dx\,dy = \int_0^2\int_0^x e^{x^2}dy\,dx = \int_0^2 x\,e^{x^2}dx = \frac{1}{2}e^{x^2}]_0^2 = \frac{1}{2}[e^4-1]$.

(La integral inicial no tiene primitiva elemental).

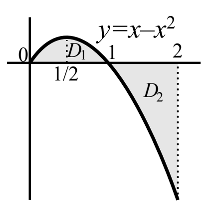


5. El recinto aparece dividido en dos regiones: $\iint_D f = \iint_{D_1} f + \iint_{D_2} f$.

$\int_0^1\int_0^{x-x^2} (x^2+2xy^2+2)\,dy\,dx + \int_1^2\int_{x-x^2}^0 (x^2+2xy^2+2)\,dy\,dx$

$= \int_0^1 [(x^2+2)(x-x^2)+\frac{2x}{3}(x-x^2)^3]\,dx - \int_1^2 [(x^2+2)(x-x^2)+\frac{2x}{3}(x-x^2)^3]\,dx$

$= \int_0^1 (2x-2x^2+x^3-\frac{x^4}{3}-2x^5+2x^6-\frac{2x^7}{3})\,dx - \int_1^2 (\cdot)\,dx = \frac{27}{70} + \frac{1249}{210} = \frac{19}{3}$.

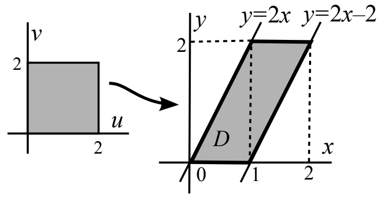


6. i) Para no hacer 2 integrales, es mejor integrar primero respecto a x:

$$\int_0^2\int_{y/2}^{y/2+1}(2x-y)^3\,dx\,dy=\int_0^2\tfrac18\big[(2x-y)^4\big]_{y/2}^{y/2+1}dy=\int_0^2 2\,dy=\boxed{4}\,.$$

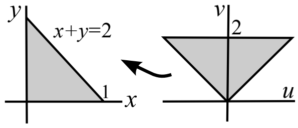
Más largo: $\int_0^1\int_0^{2x}(2x-y)^3dy\,dx+\int_1^2\int_{2x-2}^2(2x-y)^3dy\,dx$

$$=\int_0^1 4x^4dx+\int_1^2\big[4-4(x-1)^4\big]\,dx=\tfrac45+4-\tfrac45=4\,.$$



ii) Con el cambio: $\begin{matrix}u=2x-y\\ v=y\end{matrix}$, $\begin{matrix}x=\frac{u+v}{2}\\ y=v\end{matrix}$, $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1/2&1/2\\0&1\end{vmatrix}=\frac12$. $\frac12\int_0^2\int_0^2u^3du\,dv=1\cdot\big[\frac14u^4\big]_0^2=\boxed{4}\,.$

7. $\begin{cases}u=y-x\\ v=y+x\end{cases}\Leftrightarrow\begin{cases}x=(v-u)/2\\ y=(u+v)/2\end{cases}$. $J=\begin{vmatrix}-1/2&1/2\\1/2&1/2\end{vmatrix}=-\frac12$. $\begin{matrix}x=0\to u=v\\ y=0\to u=-v\\ x+y=2\to v=2\end{matrix}$



Luego, $\iint_D e^{(y-x)/(y+x)}\,dx\,dy=\frac12\int_0^2\int_{-v}^v e^{u/v}du\,dv=\int_0^2 v(e-\frac1e)\,dv=e-\frac1e\,.$

8. i) $A=\int_0^2\int_{-x^2}^x dy\,dx+\int_2^4\int_{x-6}^{-x^2+4x-2}dy\,dx=\int_0^2(x+x^2)\,dx+\int_2^4(4+3x-x^2)\,dx=12\,.$

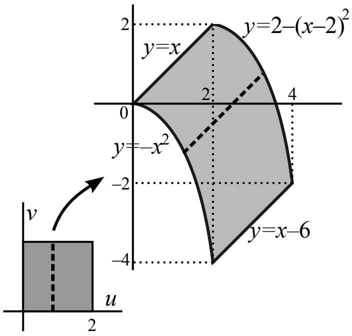
[A esto llegaríamos trabajando con la $\int$ de **R**].

ii) Un buen cambio es $\begin{matrix}x=u+v\\ y=v-u^2\end{matrix}$ pues $\begin{matrix}u=0\to y=x,\ \ u=2\to y=x-6\\ v=0\to y=-x^2,\ v=2\to(x-2)^2=2-y\end{matrix}$

$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1&1\\-2u&1\end{vmatrix}=1+2u\neq0$ en D^*, y el cambio es inyectivo en D^* pues lleva $u=a$ a rectas $y=x-a-a^2$ del plano xy, distintas para cada $a\geq0$.

[No es inyectiva en $\mathbf{R}^2$ pues, por ejemplo, $u=-1$ se transforma también en $y=x$].

Por tanto, $A=\int_0^2\int_0^2(1+2u)\,du\,dv=2\big[u+u^2\big]_0^2=12\,.$

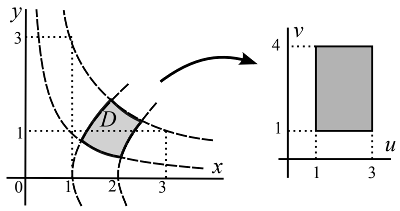


9. $u=xy$, $v=x^2-y^2\to\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}y&x\\2x&-2y\end{vmatrix}=-2(x^2+y^2)\to$

$$\iint_D(x^2+y^2)\,dx\,dy=\int_1^3\int_1^4\tfrac12\,dv\,du=3\,.$$

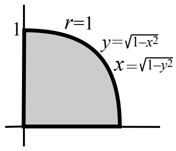
[Casualidad que coincida casi con el jacobiano].

[Despejar x,y en función de u,v es complicado].



10. a) i) Cartesianas: $\iint_D f=\int_0^1\int_0^{\sqrt{1-x^2}}x^2y\,dy\,dx=\frac12\int_0^1x^2(1-x^2)\,dx=\frac12\big[\frac13-\frac15\big]=\frac1{15}\,.$

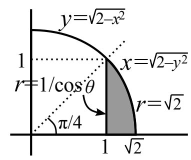
Más largo: $\int_0^1\int_0^{\sqrt{1-y^2}}x^2y\,dx\,dy=\frac13\int_0^1y\,(1-y^2)^{3/2}dy=-\frac1{15}\,(1-y^2)^{5/2}\big]_0^1=\frac1{15}\,.$



ii) En polares: $\iint_D f=\int_0^{\pi/2}\int_0^1 r^4\cos^2\theta\ \mathrm{sen}\,\theta\,dr\,d\theta=\big[\frac15r^5\big]_0^1\big[-\frac13\cos^3\theta\big]_0^{\pi/2}=\frac15\cdot\frac13=\frac1{15}\,.$

b) i) $\int_1^{\sqrt2}\int_0^{\sqrt{2-x^2}}x\,dy\,dx=\int_1^{\sqrt2}x\sqrt{2-x^2}\,dx=-\frac13(2-x^2)^{3/2}\big]_1^{\sqrt2}=\frac13$, o bien

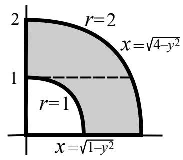
$$\int_0^1\int_1^{\sqrt{2-y^2}}x\,dx\,dy=\int_0^1\Big[\frac{x^2}{2}\Big]_1^{\sqrt{2-y^2}}dy=\int_0^1\frac{1-y^2}{2}\,dy=\frac12-\frac16=\frac13\,.$$



ii) $\int_0^{\pi/4}\int_{1/\cos\theta}^{\sqrt2}r^2\cos\theta\,dr\,d\theta=\frac13\int_0^{\pi/4}\big[2\sqrt2\,\cos\theta-\frac1{\cos^2\theta}\big]d\theta=\frac13\big[2\sqrt2\ \mathrm{sen}\,\theta-\tan\theta\big]_0^{\pi/4}=\frac13[2-1]=\frac13\,.$

c) ii) En polares: $\iint_D f\,dx\,dy=\int_0^{\pi/2}\int_1^2 r\frac{r\cos\theta}{r}\,dr\,d\theta=\big[\frac{r^2}{2}\big]_1^2\,\big[\mathrm{sen}\,\theta\big]_0^{\pi/2}=\frac32\,.$

i): $\int f\,dx=\sqrt{x^2+y^2}$, $\int_0^1\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}}f\,dx\,dy+\int_1^2\int_0^{\sqrt{4-y^2}}f\,dx\,dy=\int_0^1 1\,dy+\int_1^2[2-y]\,dy=3-\frac{4-1}{2}=\frac32\,.$

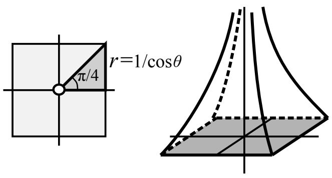


11. Se definen con límites como las impropias en **R**:

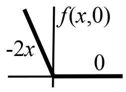
$$\iint_M\frac{dx\,dy}{(x^2+y^2)^{1/2}}\underset{\uparrow}{=}8\lim_{t\to0^+}\int_0^{\pi/4}\int_t^{1/\cos\theta}\frac rr\,dr\,d\theta=8\lim_{t\to0^+}\int_0^{\pi/4}\Big[\frac{\cos\theta}{1-\mathrm{sen}^2\theta}-t\Big]d\theta=8\log(1+\sqrt2)\,.$$

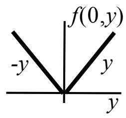
$\approx 7{,}05$

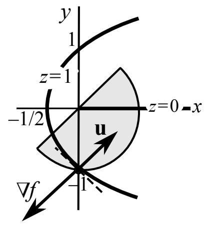
(de hecho, con el cambio deja de ser impropia)



12. $\boxed{f(x,y)=\sqrt{x^2+y^2}-x}$ a) $f(x,0)=|x|-x\,,\ f(0,y)=|y|$ no derivables $\Rightarrow\ \nexists$ **parciales** en $(0,0)\ \Rightarrow\ f$ **no es diferenciable**.







[Que f es continua en $\mathbf{R}^2$ es obvio, pues lo es la raíz para valores positivos].

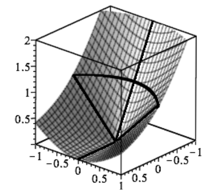
b) $f=1\ \Rightarrow\ x^2+y^2=x^2+2x+1\,,\ x=\frac{1}{2}[y^2-1]$ [parábola con $x'(-1)=-1$].

$\nabla f=\Big(\frac{x}{\sqrt{x^2+y^2}}-1\,,\frac{y}{\sqrt{x^2+y^2}}\Big)\,,\ \nabla f(0,-1)=(-1,-1)\ \Rightarrow\ \bar{u}=\Big(\frac{1}{\sqrt{2}}\,,\frac{1}{\sqrt{2}}\Big)\,,$

pues la derivada direccional el mínima en sentido opuesto al gradiente.

[En polares: $\nabla f=f_r\,\mathbf{e_r}+\frac{1}{r}f_\theta\,\mathbf{e}_\theta=(\cos\theta-1,\operatorname{sen}\theta)$].

[También se deduce del hecho de que el gradiente es perpendicular a la curva de nivel en el punto y de que apunta hacia donde crece el campo].

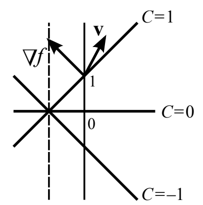


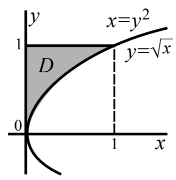
dibujo de f con Maple

c) $\displaystyle\int_{-3\pi/4}^{\pi/4}\!\int_0^1 r^2(1-\cos\theta)\,dr\,d\theta=\frac{1}{3}\big(\pi-[\operatorname{sen}\theta]_{-3\pi/4}^{\pi/4}\big)=\boxed{\frac{\pi-\sqrt{2}}{3}}$. [En cartesianas las integrales son bastante complicadas].

13. a] $\boxed{f(x,y)=\frac{y}{x+1}}=0,1,-1\ \to$ rectas $y=0,\ y=x+1,\ y=-x-1$.

$\nabla f=\big(\frac{-y}{(x+1)^2},\frac{1}{x+1}\big)\big|_{(0,1)}=(-1,1)$. $\Delta f=\frac{2y}{(x+1)^3}$. $D_{\mathbf{v}}f(0,1)=(-1,1)\cdot(\frac{3}{5},\frac{4}{5})=\frac{1}{5}$.



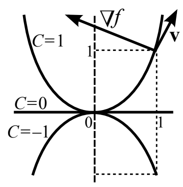


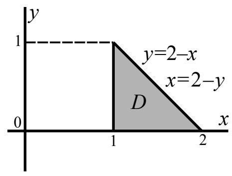
b] $\displaystyle\iint_D f=\int_0^1\!\int_{\sqrt{x}}^1\frac{y}{x+1}\,dy\,dx=\frac{1}{2}\int_0^1\frac{1-x}{x+1}\,dx=\int_0^1\Big[\frac{1}{x+1}-\frac{1}{2}\Big]dx=\ln|x+1|\,\Big]_0^1-\frac{1}{2}=\ln 2-\frac{1}{2}$.

Más corto que: $\displaystyle\int_0^1\!\int_0^{y^2}\frac{y}{x+1}\,dx\,dy=\int_0^1 y\ln(1+y^2)\,dy\underset{\text{partes}}{=}\frac{y^2}{2}\ln(1+y^2)\Big]_0^1-\int_0^1\frac{y^3+y-y}{1+y^2}\,dy=\frac{1}{2}\ln 2-\frac{1}{2}+\frac{1}{2}\ln 2$.

14. a] $\boxed{f(x,y)=\frac{y}{x^2}}=0,1,-1\ \to\ y=0,\ y=x^2,\ y=-x^2$ (parábolas).

$\nabla f=\big(\frac{-2y}{x^3},\frac{1}{x^2}\big)\big|_{(1,1)}=(-2,1)$. $\Delta f=\frac{6y}{x^4}$. $D_{\mathbf{v}}f(1,1)=(-2,1)\cdot(\frac{3}{5},\frac{4}{5})=-\frac{2}{5}$.



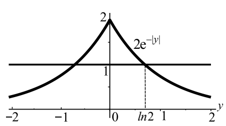


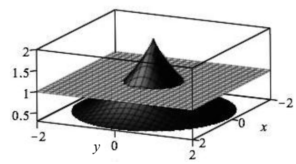
b] $\displaystyle\iint_D f=\int_1^2\!\int_0^{2-x}\frac{y}{x^2}\,dy\,dx=\frac{1}{2}\int_1^2\frac{4-4x+x^2}{x^2}\,dx=\Big[-\frac{2}{x}-2\ln|x|\Big]_1^2+\frac{1}{2}=\frac{3}{2}-2\ln 2$.

Algo más corto que: $\displaystyle\int_0^1\!\int_1^{2-y}\frac{y}{x^2}\,dx\,dy=\int_0^1 y\,\Big[1-\frac{1}{2-y}\Big]\,dy=\int_0^1\Big[y+1-\frac{2}{2-y}\Big]\,dy=\frac{1}{2}+1+2\ln 2$.

15. z positiva en el rectángulo: $\displaystyle V=\int_0^1\!\int_1^2(x^2+y)\,dy\,dx=\int_0^1 x^2\,dx+\int_1^2 y\,dy=\frac{1}{3}+\frac{4-1}{2}=\frac{11}{6}$.

16. $\boxed{g(x,y)=2\,e^{-\sqrt{x^2+y^2}}}$ a) De revolución. $g(0,y)=2e^{-|y|}\ \Rightarrow\ g_y(0,0)$ no existe $\Rightarrow\ g$ **no diferenciable** en $(0,0)$.



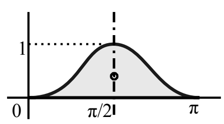


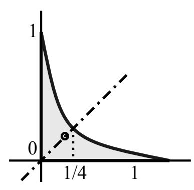
b) $2\,e^{-\sqrt{x^2+y^2}}=1\ \Leftrightarrow\ r=\sqrt{x^2+y^2}=\log 2$. Polares-cilíndricas:

$\displaystyle V=\int_0^{2\pi}\!\int_0^{\ln 2}r\,[2\,e^{-r}-1]\,dr\,d\theta=2\pi\Big[-2(r+1)\,e^{-r}-\frac{1}{2}r^2\Big]_0^{\ln 2}=2\pi\big[1-\ln 2-\frac{1}{2}(\ln 2)^2\big]$.

17. a) $\{0\le y\le\operatorname{sen}^2x,\ 0\le x\le\pi\}$. Por simetría, $\bar{x}=\frac{\pi}{2}$.

$\displaystyle\bar{y}=\frac{1}{\int_0^\pi\operatorname{sen}^2x\,dx}\int_0^\pi\!\int_0^{\operatorname{sen}^2x}y\,dy\,dx=\frac{2}{\pi}\int_0^\pi\frac{\operatorname{sen}^4x}{2}dx=\frac{3}{8}$.





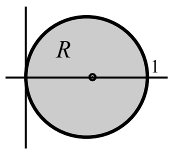
b) $\{\sqrt{x}+\sqrt{y}\le 1,\ x\le 0,\ y\ge 0\}$. $y=(1-\sqrt{x}\,)^2=1+x-2\sqrt{x}$.

$\displaystyle\bar{x}=\frac{1}{\int_0^1(1+x-2\sqrt{x})\,dx}\int_0^1\!\int_0^{1+x-2\sqrt{x}}x\,dy\,dx=6\int_0^1(x+x^2-2x^{3/2})\,dx=\frac{1}{5}=\bar{y}$, por simetría.

18. $\displaystyle M=2\int_0^{\pi/2}\!\int_0^{\cos\theta}r\cos\theta\,dr\,d\theta=\int_0^{\pi/2}(1-\operatorname{sen}^2\theta)\cos\theta\,d\theta=\frac{2}{3}$.

$\displaystyle\bar{x}=\frac{1}{M}\,2\int_0^{\pi/2}\!\int_0^{\cos\theta}r^2\cos^2\theta\,dr\,d\theta=\int_0^{\pi/2}(1-\operatorname{sen}^2\theta)^2\cos\theta\,d\theta=\frac{8}{15}$.

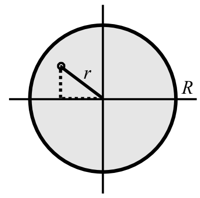
$\bar{y}=0$ por simetría de R y de su función densidad.



19. Con la distancia usual: $\displaystyle d_{\text{media}}=\frac{1}{\pi R^2}\int_0^{2\pi}\!\int_0^R r^2dr\,d\theta=\frac{2R}{3}$.

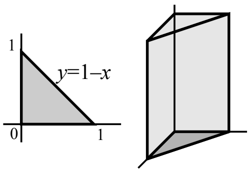
Con la nueva distancia:

$\displaystyle d_{\text{media}}=\frac{1}{\pi R^2}\int_0^{2\pi}\!\int_0^R r^2(|\cos\theta|+|\operatorname{sen}\theta|)\,dr\,d\theta=\frac{4}{\pi R^2}\int_0^{\pi/2}\!\int_0^R r^2(\cos\theta+\operatorname{sen}\theta)\,dr\,d\theta=\frac{8R}{3\pi}$.



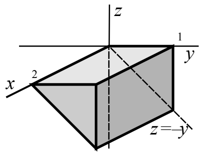
20. $\iiint_B (2x+3y+z)\,dx\,dy\,dz = 2\int_1^2 2x\,dx + \int_{-1}^1 3y\,dy + 2\int_0^1 z\,dz = 2(4-1)+0+1 = 7\,.$

21. $\iiint_V x^2\cos z\,dx\,dy\,dz = \int_0^1\int_0^{1-x}\int_0^{\pi} x^2\cos z\,dz\,dy\,dx = \int_0^1\int_0^{1-x} 0\,dy\,dx = 0\,.$



22. $\int_0^2\int_0^1\int_{-y}^0 e^y\,dz\,dy\,dx = \int_0^2\int_0^1 y\,e^y\,dy\,dx = 2\int_0^1 y\,e^y\,dy = 2[(y-1)\,e^y]_0^1 = 2\,.$

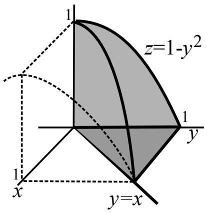
O bien: $\int_0^2\int_{-1}^0\int_{-z}^1 e^y\,dy\,dz\,dx = 2\int_{-1}^0 [e - e^{-z}]\,dz = 2\,e + 2[e^{-z}]_{-1}^0 = 2\,.$



23. Es importante elegir un buen orden de integración:

$$\int_0^1\int_0^y\int_0^{1-y^2} e^{-z}\,dz\,dx\,dy = \tfrac12\int_0^1 y(1-e^{y^2-1})\,dy = \tfrac12[y^2 - e^{y^2-1}]_0^1 = \boxed{\tfrac{1}{2e}}\,.$$

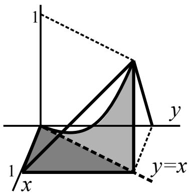
[Por ejemplo, con $\int_0^1\int_x^1\int_0^{1-y^2} e^{-z}\,dz\,dy\,dx$ aparece $\int_x^1 e^{y^2-1}dy$ no calculable].

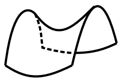


24. $\iiint_V xy^2z^3\,dx\,dy\,dz = \int_0^1\int_0^x\int_0^{xy} xy^2z^3\,dz\,dy\,dx$

$= \frac14\int_0^1\int_0^x x^5y^6\,dy\,dx = \frac{1}{28}\int_0^1 x^{12}dx = \frac{1}{364}\,.$

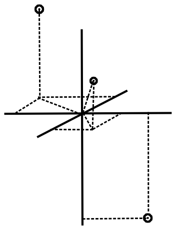
[$z = xy$ es un ‘paraboloide hiperbólico’ (silla de montar)].





25.

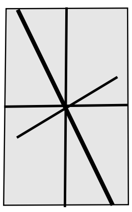
	cartesianas	polares	esféricas
P	$(0,2,-4)$	$(2,\frac{\pi}{2},-4)$	$(2\sqrt5,\frac{\pi}{2},\pi-\arctan\frac12)$
Q	$(-2,-2\sqrt3,3)$	$(4,\frac{4\pi}{3},3)$	$(5,\frac{4\pi}{3},\arctan\frac43)$
R	$\left(\frac{\sqrt2}{2},\frac{\sqrt2}{2},1\right)$	$(1,\frac{\pi}{4},1)$	$(\sqrt2,\frac{\pi}{4},\frac{\pi}{4})$



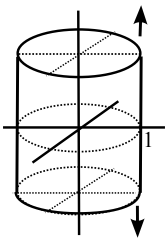
26.

$A=\{(x,y,z):\ x=0,\ z=-2y\}$	$\{x^2+y^2=1\}$	$\{x^2+y^2\le z^2\le 1-x^2-y^2\}$
$\{\theta=\frac{\pi}{2},\ z=-2r$ ó $\theta=-\frac{\pi}{2},\ z=2r\}$	$B=\{(r,\theta,z):\ r=1\}$	$\{r\le z\le\sqrt{1-r^2}\}$
$\{\theta=\frac{\pi}{2},\ \phi=\frac{5\pi}{6}$ ó $\theta=-\frac{\pi}{2},\ \phi=\frac{\pi}{6}\}$	$\{\rho\,\mathrm{sen}\,\theta=1\}$	$C=\{(\rho,\theta,\phi):\ \phi\le\frac{\pi}{4},\ \rho\le 1\}$

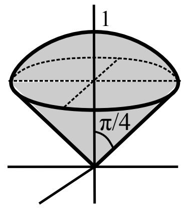
recta por el origen



cilindro vertical



volumen entre cono y esfera

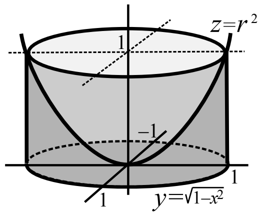


27. a) Claramente las coordenadas adecuadas son las cilíndricas:

$$\int_0^{2\pi}\int_0^1\int_0^{r^2} rz\,dz\,dr\,d\theta = 2\pi\,\tfrac12\int_0^1 r^5dr = \boxed{\tfrac{\pi}{6}}\,.$$

[Largo en cartesianas: $\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{x^2+y^2} z\,dz\,dy\,dx = 2\int_0^1\int_0^{\sqrt{1-x^2}}(x^4+2x^2y^2+y^4)\,dy\,dx$

$= \frac{2}{15}\int_0^1 (8x^4+4x^2+3)\sqrt{1-x^2}\,dx \underset{x=\mathrm{sen}\,t\uparrow}{=} \frac{2}{15}\int_0^{\pi/2}(8\,\mathrm{sen}^4t+4\,\mathrm{sen}^2t+3)\cos^2t\,dt = \cdots$].

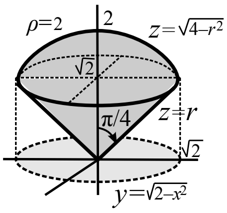


b) El cono $z=\sqrt{x^2+y^2}$ y la esfera $x^2+y^2+z^2=4$ piden hacer la integral en esféricas

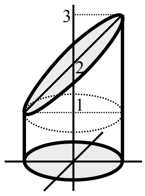
$$\iiint_V z = 2\pi\int_0^{\pi/4}\int_0^2 \rho^3\,\mathrm{sen}\,\phi\cos\phi\,d\rho\,d\phi = \pi\,[\tfrac14\rho^4]_0^2[\,\mathrm{sen}^2\phi]_0^{\pi/4} = 2\pi\,.$$

En cilíndricas: $2\pi\int_0^{\sqrt2}\int_r^{\sqrt{4-r^2}} rz\,dz\,dr = 2\pi\int_0^{\sqrt2} r(2-r^2)\,dr = 2\pi\,[r^2-\frac{r^4}{4}]_0^{\sqrt2} = 2\pi\,.$

En cartesianas: $\int_{-\sqrt2}^{\sqrt2}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z\,dz\,dy\,dx = \int_{-\sqrt2}^{\sqrt2}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}(2-x^2-y^2)\,dy\,dx = \cdots$



28. a) $V=\iiint_V 1=\int_0^{2\pi}\int_0^1\int_0^{r\,\text{sen}\,\theta+2} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1 (r^2\,\text{sen}\,\theta+2r)\,dr\,d\theta=\int_0^{2\pi}\big[\frac{\text{sen}\,\theta}{3}+1\big]d\theta=2\pi$.

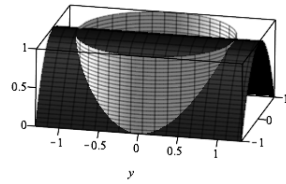


Peor en cartesianas: $V=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{y+2} dz\,dy\,dx=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(y+2)\,dy\,dx=4\int_{-1}^{1}\sqrt{1-x^2}\,dx$

$=[\text{par},\ x=\text{sen}\,t\,]=8\int_0^{\pi/2}\cos^2 t\,dt=4\int_0^{\pi/2}(1+\cos 2t)\,dt=2\pi$.

[Sin integrar: volumen de cilindro de altura 1 más medio volumen de cilindro de altura 2].

b) Sobre $x^2+y^2\le 1$ la gráfica de $z=1-x^2$ está por encima del plano $z=0$:



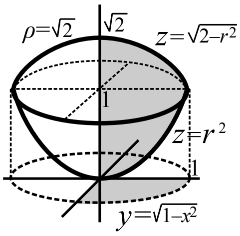
$$V=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2} dz\,dy\,dx=2\int_{-1}^{1}(1-x^2)^{3/2}dx\underset{x=\text{sen}\,t}{=}4\int_0^{\pi/2}\cos^4 t\,dt$$

$$=\int_0^{\pi/2}\big(1+2\cos 2t+\tfrac{1+\cos 4t}{2}\big)\,dt=\tfrac{3\pi}{4}\ .$$

O, más corto, en cilíndricas:

$$V=\int_0^{2\pi}\int_0^1\int_0^{1-r^2\cos^2\theta} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1(r-r^3\cos^2\theta)dr\,d\theta=\int_0^{2\pi}\big(\tfrac{1}{2}-\tfrac{\cos^2\theta}{4}\big)\,d\theta=\int_0^{2\pi}\tfrac{3-\cos 2\theta}{8}\,d\theta=\tfrac{3\pi}{4}\ .$$

c) Cilíndricas: $V=2\pi\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr=2\pi\int_0^1\big(r\sqrt{2-r^2}-r^3\big)\,dr=\frac{\pi}{6}(8\sqrt{2}-7)$.



O bien: $V=\int_0^{2\pi}\int_0^1\int_0^{\sqrt{z}} r\,dr\,dz\,d\theta+\int_0^{2\pi}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-r^2}} r\,dr\,dz\,d\theta$

$=\pi\int_0^1(\sqrt{z}\,)^2dz+\pi\int_1^{\sqrt{2}}(\sqrt{2-z^2}\,)^2dz$ [a esto se llegaría utilizando fórmulas de una variable]

$=\frac{\pi}{2}+2\pi(\sqrt{2}-1)-\frac{\pi}{3}(2\sqrt{2}-1)=\frac{\pi}{6}(8\sqrt{2}-7)$.

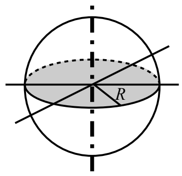
En esféricas: $V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}}\rho^2\,\text{sen}\,\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cos\phi/\,\text{sen}^2\phi}\rho^2\,\text{sen}\,\phi\,d\rho\,d\phi\,d\theta$
$[\,\rho\cos\phi=\rho^2\,\text{sen}^2\phi\,(\text{sen}^2\theta+\cos^2\theta)\,]$

$$=2\pi\,\tfrac{2\sqrt{2}}{3}\big[-\cos\phi\big]_0^{\pi/4}+\tfrac{2\pi}{3}\int_{\pi/4}^{\pi/2}\tfrac{\cos^3\phi}{\text{sen}^5\phi}\,d\phi=\tfrac{4\pi}{3}(\sqrt{2}-1)+\tfrac{2\pi}{3}\int_{1/\sqrt{2}}^{1}\tfrac{1-t^2}{t^5}\,dt=\tfrac{\pi}{6}(8\sqrt{2}-7)\ .$$

En cartesianas: $V=4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx=\cdots$ (difíciles integrales).

29. $\iiint_V f\,dx\,dy\,dz=\int_0^{2\pi}\int_0^{\pi}\int_b^a\frac{\rho^2\,\text{sen}\,\phi}{\rho^3}d\rho\,d\phi\,d\theta=2\pi\big[\log\rho\big]_b^a\int_0^{\pi}\text{sen}\,\phi\,d\phi=4\pi\log\frac{a}{b}$.

30. $M=\frac{4}{3}\pi R^3\rho^*$, con ρ^* densidad constante.



$$I_z=\iiint_V(x^2+y^2)\,\rho^*dx\,dy\,dz=\rho^*\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho^4\,\text{sen}^3\phi\,d\rho\,d\phi\,d\theta$$

$$=\tfrac{2\pi R^5\rho^*}{5}\int_0^{\pi}(1-\cos^2\phi)\,\text{sen}\,\phi\,d\phi=\tfrac{8\pi R^5\rho^*}{15}=\tfrac{2MR^2}{5}\ .$$