Symmetries of Discrete Dynamical Systems Involving Two Species

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Abstract

The Lie point symmetries of a coupled system of two nonlinear differential-difference equations are investigated. It is shown that in special cases the symmetry group can be infinite dimensional, in other cases up to 10 dimensional. The equations can describe the interaction of two long molecular chains, each involving one type of atoms.

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The purpose of this article is to perform a symmetry analysis of a system of two coupled differential-difference equations of the form

\[
\begin{align*}
E_1 &= \ddot{u}_n - F_n(t, u_{n-1}, u_n, u_{n+1}, v_{n-1}, v_n, v_{n+1}) = 0, \\
E_2 &= \ddot{v}_n - G_n(t, u_{n-1}, u_n, u_{n+1}, v_{n-1}, v_n, v_{n+1}) = 0.
\end{align*}
\] (1.1)

The dots denote time derivatives. The discrete variable \(n\) plays the role of a space variable; it labels positions along a one dimensional lattice. The functions \(F_n\) and \(G_n\) represent interactions e.g. between different atoms along a double chain of molecules (see Fig.1). The functions \(F_n\) and \(G_n\) are a priori unspecified; our aim is to classify equations of the type (1.1) according to the Lie point symmetries that they allow. The interactions in such a model depend on up to six neighbouring particles. For instance, we can interpret \(u_n\) and \(v_n\) as deviations from equilibrium positions of two different types of atoms, say type \(U\) and type \(V\). The accelerations \(\ddot{u}_n\) and \(\ddot{v}_n\) depend on the deviations \(u\) and \(v\) of both types of atoms at the neighbouring sites \(n - 1, n\) and \(n + 1\). We do not restrict to two-body forces, nor do we impose translational invariance for the chain. We do however assume there is no dissipation, i.e. system (1.1) does not involve first derivatives with respect to time.

Such differential-difference equations typically arise when modeling phenomena in molecular physics, biophysics, or simply coupled oscillations in classical mechanics [1,2,3].
A recent article [4] was devoted to a similar problem, but was concerned with a single species, i.e. one dependent variable $u_n(t)$. The approach adopted here is similar to that of Ref.4. Thus, we shall consider only symmetries acting on the continuous variables $t, u_n$ and $v_n$. Transformations of the discrete variable $n$ must then be studied separately.

Several different treatments of Lie symmetries of difference and differential-difference equations exist in the literature [4-13]. The one adopted in this article is that of Ref.4,5,6. It has been called the “intrinsic method”, makes use of a Lie algebraic approach and is entirely algorithmic. The Lie algebra of the symmetry group, the “symmetry algebra” for short, is realized by vector fields of the form

$$\hat{X} = \tau(t, u_n, v_n) \partial_t + \phi_n(t, u_n, v_n) \partial_{u_n} + \psi_n(t, u_n, v_n) \partial_{v_n}. \quad (1.2)$$

The algorithm for finding the functions $\tau, \phi_n$ and $\psi_n$ in (1.2) is to construct the appropriate prolongation $pr\hat{X}$ of $\hat{X}$ (see Ref.4,5,6 and Section II below) and to impose that it should annihilate the studied system of equations on their solution set

$$pr\hat{X}E_1|_{E_1=E_2=0} = 0, \quad pr\hat{X}E_2|_{E_1=E_2=0} = 0. \quad (1.3)$$

Our first step is to find and classify all interactions $(F_n, G_n)$ for which the system (1.1) allows at least a one dimensional symmetry algebra. The next step is to specify the interactions further and to find
all those that allow a higher dimensional, possibly infinite dimensional, symmetry algebra.

As in previous articles [4,14] our classification will be up to conjugacy under a group of “allowed transformations”. These are fiber preserving locally invertible point transformations

\[ u_n = \Omega_n(\tilde{u}_n, \tilde{v}_n, \tilde{t}), \quad v_n = \Gamma_n(\tilde{u}_n, \tilde{v}_n, \tilde{t}), \quad t = t(\tilde{t}) \quad (1.4) \]

that preserve the form of equations (1.1), but not necessarily the functions \( F_n \) and \( G_n \) (they go into new functions \( \tilde{F}_n \) and \( \tilde{G}_n \) of the new arguments).

Throughout the article we assume that both \( F_n \) and \( G_n \) depend on at least one of the quantities \( u_{n-1}, u_{n+1}, v_{n-1}, v_{n+1} \), so that nearest neighbours are genuinely involved. In the bulk of the article the interaction is assumed to be nonlinear.

In Section II we formulate the problem, establish the general form of the elements of the symmetry algebra and present the determining equations for the symmetries. We also derive the “allowed transformations” under which we classify the interactions and their symmetries. Section III is devoted to a classification of interactions \( F_n, G_n \) allowing at least a one dimensional symmetry algebra. Ten classes of such interactions exist, each involving 2 arbitrary functions of 6 variables. In Section IV we study higher dimensional symmetry algebras and introduce an important restriction. We first prove that 4 equivalence
classes of symmetry algebras isomorphic to \( sl(2, \mathbb{R}) \) exist. Then we restrict to just one of them, \( sl(2, \mathbb{R})_1 \) generating a gauge group acting only on the fields \( u_n \) and \( v_n \) (in a global, coordinate independent manner). We describe all symmetry algebras, containing the chosen \( sl(2, \mathbb{R}) \) as a subalgebra. In Section V we obtain the invariant interactions for all algebras containing \( sl(2, \mathbb{R})_1 \). The results are summed up and discussed in Section VI where we also outline future work to be done.

II FORMULATION OF THE PROBLEM

To find the Lie point symmetries of the system (1.1) we write the second prolongation of the vector field (1.2) in the form [4, 5, 6]:

\[
\text{pr}^{(2)} \tilde{X} = \tau(t, u_n, v_n) \partial_t + \sum_{k=n-1}^{n+1} \phi_k(t, u_n, v_n) \partial_{u_k} + \sum_{k=n-1}^{n+1} \psi_k(t, u_n, v_n) \partial_{v_k} + \phi_{tt}^u \partial_{\ddot{u}_n} + \psi_{tt}^v \partial_{\ddot{v}_n},
\]

(2.1)

where \( D_t \) is the total time derivative. The determining equations for the symmetries are obtained by requiring that eq.(1.3) be satisfied. The obtained equations will involve terms like \( \dot{u}_k, \dot{v}_k \) and \( \ddot{u}_k \ddot{v}_l \). The coefficients of each linearly independent term must vanish and this
provides 16 linear differential equations that are easy to solve and do not involve the interaction functions $F_n, G_n$. The result is that an element $\hat{X}$ of the symmetry algebra must have the form

$$\hat{X} = \tau(t)\partial_t + \left[\left(\frac{\dot{\tau}}{2} + a_n\right) u_n + b_n v_n + \lambda_n(t)\right]\partial_{a_n}$$

$$+ \left[c_n u_n + \left(\frac{\dot{\tau}}{2} + d_n\right) v_n + \mu_n(t)\right]\partial_{b_n},$$

where the dots denote time derivatives. The functions $\tau(t), \lambda_n(t), \mu_n(t), a_n, b_n, c_n$ and $d_n$ satisfy the two remaining determining equations, namely

$$\frac{\ddot{\tau}}{2} u_n + \ddot{\lambda}_n + \left(a_n - \frac{3}{2} \dot{\tau}\right) F_n + b_n G_n - \tau F_{n,t}$$

$$- \sum_{k=n-1}^{n+1} F_{n,u_k} \left[\left(\frac{\dot{\tau}}{2} + a_k\right) u_k + b_k v_k + \lambda_k(t)\right] = 0,$$

$$\frac{\ddot{\tau}}{2} v_n + \ddot{\mu}_n + \left(d_n - \frac{3}{2} \dot{\tau}\right) G_n + c_n F_n - \tau G_{n,t}$$

$$- \sum_{k=n-1}^{n+1} G_{n,u_k} \left[\left(\frac{\dot{\tau}}{2} + a_k\right) u_k + b_k v_k + \lambda_k(t)\right] = 0.$$
to simplify this vector. The second step is to find interactions $F_n$ and $G_n$ compatible with such a symmetry.

Substituting (1.4) into eq.(1.1) and requiring that the form of these two equations be preserved, we find that the allowed transformations are quite restricted, namely

$$
\begin{pmatrix}
    \tilde{u}_n(t) \\
    \tilde{v}_n(t)
\end{pmatrix} = \begin{pmatrix}
    Q_n & R_n \\
    S_n & T_n
\end{pmatrix} \tilde{t}^{-1/2} \begin{pmatrix}
    \tilde{u}_n(\tilde{t}) \\
    \tilde{v}_n(\tilde{t})
\end{pmatrix} + \begin{pmatrix}
    \alpha_n(t) \\
    \beta_n(t)
\end{pmatrix},
$$

(2.6)

$$
\tilde{t} = \tilde{t}(t), \quad \frac{d\tilde{t}}{dt} \neq 0.
$$

The entries $Q_n$, $R_n$, $S_n$ and $T_n$ are independent of $t$; $\tilde{t}(t)$ is an arbitrary locally invertible function of $t$; $\alpha_n$, $\beta_n$ are arbitrary functions of $n$ and $t$, and the matrix

$$
M_n = \begin{pmatrix}
    Q_n & R_n \\
    S_n & T_n
\end{pmatrix}, \quad \text{det} \ M_n \neq 0
$$

(2.7)

is nonsingular.

It will be convenient to use a short-hand notation for the vector field $X_n$ of eq.(2.3), namely

$$
\left\{ \tau(t), \lambda_n(t), \lambda_n(t) \right\}, \quad A_n = \begin{pmatrix}
    a_n & b_n \\
    c_n & d_n
\end{pmatrix}.
$$

(2.8)

If we perform an allowed transformation (2.6), then eq.(1.1) goes into an equation of the same form, with $F_n$ and $G_n$ replaced by

$$
\begin{pmatrix}
    \tilde{F}_n \\
    \tilde{G}_n
\end{pmatrix} = \tilde{t}^{-3/2} M_n^{-1} \left[ \begin{pmatrix}
    F_n \\
    G_n
\end{pmatrix} - \begin{pmatrix}
    \tilde{\alpha}_n \\
    \tilde{\beta}_n
\end{pmatrix} \right]

+ \left( \frac{1}{2} \tilde{t}^{-3} - \frac{3}{4} \tilde{t}^{-4} \right) \begin{pmatrix}
    \tilde{u}_n \\
    \tilde{v}_n
\end{pmatrix},
$$

(2.9)
where $\tilde{F}_n$ and $\tilde{G}_n$ are functions of the new variables.

The vector field characterized by the triplet (2.3) goes into a new one of the same form

$$\left\{ \tilde{\tau}(\tilde{t}), \tilde{A}_n, \left( \begin{array}{c} \tilde{\lambda}_n(\tilde{t}) \\ \tilde{\mu}_n(\tilde{t}) \end{array} \right) \right\}$$  \hspace{1cm} (2.10)

with

$$\tilde{\tau}(\tilde{t}) = \tau(t(\tilde{t})) \dot{\tilde{t}},$$
$$\tilde{A}_n = M_n^{-1} A_n M_n$$
$$\left( \begin{array}{c} \tilde{\lambda}_n(\tilde{t}) \\ \tilde{\mu}_n(\tilde{t}) \end{array} \right) = M_n^{-1} \dot{\tilde{t}}^{1/2} \left[ (A_n + \frac{\tau}{2}) \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} - \tau \begin{pmatrix} \dot{\alpha}_n \\ \dot{\beta}_n \end{pmatrix} + \begin{pmatrix} \lambda_n \\ \mu_n \end{pmatrix} \right].$$ \hspace{1cm} (2.11)

We shall use the allowed transformations to simplify the vector field, rather than the equation itself.

### III SYSTEMS WITH ONE-DIMENSIONAL SYMMETRY GROUPS

Let us now assume that the system (1.1) has at least a one dimensional symmetry group, generated by a vector field of the type (2.3). Using allowed transformations (2.6), we take $\hat{X}$ into one of 10 inequivalent classes.

Indeed, for $\tau \neq 0$ we can choose the function $\tilde{t}(t)$ so as to transform $\tau(t)$ into $\tau = 1$, the functions $\alpha_n(t)$ and $\beta_n(t)$ so as to annul $\lambda_n(t)$ and $\mu_n(t)$ and the matrix $M_n$ so as to take $A_n$ into its canonical form.

For $\tau = 0$ the standardized form of $\hat{X}$ depends on the rank of the matrix $A_n$. For rank $A_n = 2$ we can again transform $\lambda_n$ and $\mu_n$.
into $\lambda_n = \mu_n = 0$ and take $A_n$ into one of 3 canonical forms. For rank $A_n = 1$ only one of the functions $\lambda_n$ or $\mu_n$ can be annulled. We choose it to be $\lambda_n(t) = 0$. Then $A_n$ can be taken into one of the two standard matrices of rank 1 in $\mathbb{R}^{2 \times 2}$. For rank $A_n = 0$ both $\lambda_n(t)$ and $\mu_n(t)$ survive.

We thus obtain 10 mutually inequivalent one dimensional symmetry algebras, listed below. The statement now is that any single vector field $\hat{X}$ of the form (2.3) can be transformed by an allowed transformation into precisely one of these vector fields.

The next step is to determine the interactions for which a one dimensional symmetry group exists. To do this, we run through the canonical vector fields just obtained, substitute the corresponding $\tau (= 1 \text{ or } 0)$, $A_n$, $\lambda_n(t)$ and $\mu_n(t)$ into eqs. (2.4) and (2.5) and solve these equations for $F_n$ and $G_n$.

Following this procedure, we arrive at the following list of interactions and their one dimensional symmetry algebras.
\( A_{1,1} \)
\[
\dot{X} = \partial_t + a_n u_n \partial_{u_n} + d_n v_n \partial_{v_n}, \\
F_n = e^{a_n t} f_n(\xi_k, \eta_k), \\
G_n = e^{d_n t} g_n(\xi_k, \eta_k), \\
\xi_k = u_k e^{-a_k t}, \quad \eta_k = v_k e^{-d_k t}, \\
k = n - 1, n, n + 1.
\]

\( A_{1,2} \)
\[
\dot{X} = \partial_t + (a_n u_n + v_n) \partial_{u_n} + a_n v_n \partial_{v_n}, \\
F_n = e^{a_n t} \left[ f_n(\xi_k, \eta_k) + t g_n(\xi_k, \eta_k) \right], \\
G_n = e^{a_n t} g_n(\xi_k, \eta_k), \\
\xi_k = (u_k - t v_k) e^{-a_k t}, \quad \eta_k = v_k e^{-a_k t}, \\
k = n - 1, n, n + 1.
\]

\( A_{1,3} \)
\[
\dot{X} = \partial_t + (a_n u_n + b_n v_n) \partial_{u_n} + (-b_n u_n + a_n v_n) \partial_{v_n}, \quad b_n > 0, \\
\begin{pmatrix} F_n \\ G_n \end{pmatrix} = e^{a_n t} \begin{pmatrix} \cos b_n t & \sin b_n t \\ -\sin b_n t & \cos b_n t \end{pmatrix} \begin{pmatrix} f_n(\xi_k, \eta_k) \\ g_n(\xi_k, \eta_k) \end{pmatrix}, \\
\xi_k = r_k e^{-a_k t}, \quad \eta_k = \theta_k + b_k t, \\
u_k = r_k \cos \theta_k, \quad v_k = r_k \sin \theta_k, \\
k = n - 1, n, n + 1.
\]

\( A_{1,4} \)
\[
\dot{X} = a_n u_n \partial_{u_n} + d_n v_n \partial_{v_n}, \quad |a_n| \geq |d_n|, \\
F_n = u_n f_n(\xi_k, \eta_k, t), \\
G_n = v_n g_n(\xi_k, \eta_k, t), \\
\xi_k = u_n^{a_n} u_n^{-a_n}, \quad \eta_k = v_n^{a_n} u_n^{-d_k}, \\
k = n - 1, n, n + 1, \quad \alpha = n - 1, n + 1.
\]
\( A_{1.5} \quad \dot{X} = (a_n u_n + v_n) \partial_{a_n} + a_n v_n \partial_{v_n}, \quad a_n \neq 0, \)

\( F_n = v_n f_n(\eta_k, \xi_k, t) + v_n \ln(v_n) g_n(\eta_k, \xi_k, t), \)

\( G_n = a_n v_n g_n(\eta_k, \xi_k, t), \)

\( \xi_k = a_k \frac{u_k}{v_k} - \ln(v_k), \quad \eta_k = v_n^{a_n} v_n^{-a_n}, \)

\( k = n - 1, n, n + 1, \quad \alpha = n - 1, n + 1. \)

\( A_{1.6} \quad \dot{X} = v_n \partial_{a_n}, \)

\( F_n = f_n(v_k, \xi_k, t) + u_n g_n(v_k, \xi_k, t), \)

\( G_n = v_n g_n(v_k, \xi_k, t), \)

\( \xi_k = -v_n u_n + v_n u_k, \)

\( k = n - 1, n, n + 1, \quad \alpha = n - 1, n + 1. \)

\( A_{1.7} \quad \dot{X} = (a_n u_n + b_n v_n) \partial_{a_n} + (-b_n u_n + a_n v_n) \partial_{v_n}, \quad b_n > 0, \)

\[
\begin{pmatrix}
F_n \\
G_n
\end{pmatrix} = e^{-\frac{\theta_n}{b_n}} \begin{pmatrix}
\cos \theta_n & -\sin \theta_n \\
\sin \theta_n & \cos \theta_n
\end{pmatrix} \begin{pmatrix}
f_n(\xi_k, \eta_k, t) \\
g_n(\xi_k, \eta_k, t)
\end{pmatrix}
\]

\( \xi_k = r_k^b e^{a_k \theta_n}, \quad \eta_k = b_n \theta_n - b_n \theta_n, \)

\( u_k = r_k \cos \theta_k, \quad v_k = r_k \sin \theta_k, \)

\( k = n - 1, n, n + 1, \quad \alpha = n - 1, n + 1. \)

\( A_{1.8} \quad \dot{X} = a_n u_n \partial_{a_n} + \mu_n(t) \partial_{v_n}, \quad \mu_n \neq 0, \)

\( F_n = u_n f_n(\eta_k, \xi_k, t), \)

\( G_n = \frac{\dot{\mu}_n}{\mu_n} v_n + g_n(\eta_k, \xi_k, t), \)

\( \eta_k = \mu_n v_n - \mu_n v_n, \quad \xi_k = u_k e^{-\frac{2k\mu_n}{\mu_n}}, \)

\( k = n - 1, n, n + 1, \quad \alpha = n - 1, n + 1. \)
\( A_{1,9} \)
\[
\dot{X} = v_n \partial_{u_n} + \mu_n(t) \partial_{v_n}, \quad \mu_n \neq 0,
\]
\[
F_n = \frac{1}{2} \frac{\dot{\mu}_n}{\mu_n^2} v_n^2 + v_n g_n(\eta_{\alpha}, \eta_n, \xi_{\alpha}, t) + f_n(\eta_{\alpha}, \eta_n, \xi_{\alpha}, t),
\]
\[
G_n = \frac{\dot{\mu}_n}{\mu_n} v_n + \mu_n g_n(\eta_{\alpha}, \eta_n, \xi_{\alpha}, t),
\]
\[
\eta_{\alpha} = \mu_n^2 u_n + \frac{1}{2} \mu_{\alpha} v_n^2 - \mu_n v_n v_{\alpha}, \quad \xi_{\alpha} = \mu_{\alpha} v_n - \mu_n v_{\alpha},
\]
\[
\eta_n = \mu_n u_n - \frac{1}{2} v_n^2, \quad \alpha = n - 1, n + 1.
\]

\( A_{1,10} \)
\[
\dot{X} = \lambda_n(t) \partial_{u_n} + \mu_n(t) \partial_{v_n}, \quad \lambda_n, \ \mu_n \neq 0,
\]
\[
F_n = \frac{\dot{\lambda}_n}{\lambda_n} u_n + f_n(\eta_k, \xi_{\alpha}, t),
\]
\[
G_n = \frac{\dot{\mu}_n}{\mu_n} u_n + g_n(\eta_k, \xi_{\alpha}, t),
\]
\[
\xi_{\alpha} = \lambda_n u_{\alpha} - \lambda_{\alpha} u_n, \quad \eta_k = \mu_k u_n - \lambda_n v_k,
\]
\[
k = n - 1, n, n + 1, \quad \alpha = n - 1, n + 1.
\]

We mention that the variables \( \xi_k \) and \( \eta_k \) are to be taken exactly as above. For instance \( \xi_{n+1} \) is not an upshift of \( \xi_n \).

The above results are summed up quite simply. Namely, the existence of a one dimensional symmetry algebra restricts the interaction terms \( F_n \) and \( G_n \) to two arbitrary functions of 6 variables, rather than the original 7 variables. The algebras \( A_{1,1}, A_{1,2} \) and \( A_{1,3} \) involve time translations. Hence, the time dependence in these cases is restricted: \( F_n \) and \( G_n \) depend on time explicitly and via invariant variables \( \xi_k \) and \( \eta_k \) that in turn depend explicitly on \( t \). The algebras \( A_{1,4}, \ldots, A_{1,10} \)
correspond to gauge transformations: the group transformations act on dependent variables only. The time variable figures in the arbitrary functions.

IV HIGHER DIMENSIONAL SYMMETRY ALGEBRAS

IV.1 General strategy

The commutator of two symmetry operators (2.3) is an operator \[ X_3 = [X_1, X_2] \] of the same form, satisfying
\[
\tau_3 = \tau_1 \dot{\tau}_2 - \tau_2 \dot{\tau}_1, \\
A_{n,3} = - [A_{n,1}, A_{n,2}], \\
\begin{pmatrix}
\lambda_{n,3} \\
\mu_{n,3}
\end{pmatrix} = \tau_1 \begin{pmatrix}
\dot{\lambda}_{n,2} \\
\dot{\mu}_{n,2}
\end{pmatrix} - \tau_2 \begin{pmatrix}
\dot{\lambda}_{n,1} \\
\dot{\mu}_{n,1}
\end{pmatrix} \\
- \left( A_{n,1} + \frac{\dot{\tau}_1}{2} \right) \begin{pmatrix}
\lambda_{n,2} \\
\mu_{n,2}
\end{pmatrix} + \left( A_{n,2} + \frac{\dot{\tau}_2}{2} \right) \begin{pmatrix}
\lambda_{n,1} \\
\mu_{n,1}
\end{pmatrix}. \tag{4.1}
\]

To obtain a finite dimensional Lie algebra of symmetry operators we see that the “differential components” \( \tau_i(t) \partial_t \) must form a Lie algebra \( L_d \), the “matrix components” \( A_{n,i} \) must also form a Lie algebra \( L_m \), homomorphic to \( L_d \). Moreover, eq.(4.1) shows that the “functional components” \( (\lambda_{n,i}(t), \mu_{n,i}(t)) \) must satisfy certain cohomology conditions.

The algebra of diffeomorphisms of \( \mathbb{R}^1 \), \{ \( \tau(t) \partial_t \) \} has only 3 mutually nondiffeomorphic finite dimensional subalgebras, namely \( sl(2, \mathbb{R}) \) and its subalgebras, realised e.g. as
\[
\{ \partial_t, t \partial_t, t^2 \partial_t \} , \{ \partial_t, t \partial_t \} , \text{ and } \{ \partial_t \}, \tag{4.2}
\]
respectively.

For \( n \) fixed the matrices \( A_n \) generate the Lie algebra of \( gl(2, \mathbb{R}) \). However, since the dependence on \( n \) is arbitrary, an unlimited number of copies of \( gl(2, \mathbb{R}) \) and its subalgebras is available.

We shall not perform a complete classification of possible symmetry algebras here. Instead, we shall first concentrate on \( sl(2, \mathbb{R}) \) symmetry algebras and show that, up to allowed transformations, four different \( sl(2, \mathbb{R}) \) symmetry algebras can be constructed. We then consider just one of these four and study its extensions to higher dimensional Lie algebras.

**IV.2 Equivalence classes of \( sl(2, \mathbb{R}) \) symmetry algebras**

Since \( sl(2, \mathbb{R}) \) is a simple Lie algebra, it has no ideals. Hence a homomorphism between \( sl(2, \mathbb{R}) \) algebras is either an isomorphism, or one of the algebras is mapped onto zero. Correspondingly we have 3 possibilities to explore, we shall call them \( sl(2, \mathbb{R})_d \), \( sl(2, \mathbb{R})_m \) and \( sl(2, \mathbb{R})_c \) (where \( d \) stands for “differential”, \( m \) for “matrix” and \( c \) for “combined”).

1. **The algebra \( sl(2, \mathbb{R})_d \)**

   We have a priori
Using allowed transformations we transform \( \lambda_n \to 0, \mu_n \to 0 \). The commutation relation \([X_1, X_2] = X_1\) then requires \( \dot{\rho}_n = \dot{\sigma}_n = 0 \).

A further allowed transformation (2.6) with \( \bar{t}(t) = t, M_n = I \) and \((\alpha_n, \beta_n)\) constant will not change \( X_1 \), but take \( \rho_n \to 0, \sigma_n \to 0 \) (while leaving \( \lambda_n = \mu_n = 0 \)). The commutation relations \([X_2, X_3] = X_3\) and \([X_1, X_3] = 2X_2\) then imply \( \omega_n = \kappa_n = 0 \).

2. The algebra \( sl(2, \mathbb{R})_m \)

A priori we have

\[
\begin{align*}
X_1 &= b_n v_n \partial u_n + \lambda_n(t) \partial u_n + \mu_n(t) \partial v_n, \\
X_2 &= a_n (u_n \partial u_n - v_n \partial v_n) + \rho_n(t) \partial u_n + \sigma_n(t) \partial v_n, \\
X_3 &= c_n u_n \partial v_n + \omega_n(t) \partial u_n + \kappa_n(t) \partial v_n.
\end{align*}
\]

The structure constants cannot depend on \( n \), so the commutation relations imply

\[
a_n = a, \quad b_n c_n = bc.
\]

Given that the product \( b_n c_n \) does not depend on \( n \), we can use an allowed transformation to take \( b_n \to b, c_n \to c \). A further allowed transformation will take \( \rho_n \to 0, \sigma_n \to 0 \). The commutation relations then imply \( \lambda_n = \mu_n = 0 \) and \( \omega_n = \kappa_n = 0 \).

3. The combined algebra \( sl(2, \mathbb{R})_c \)
In view of the above results we can write a “combined” algebra as

\[ X_1 = \partial_t + \alpha v_n \partial u_n + \xi_n \partial u_n + \eta_n \partial v_n, \quad \alpha \neq 0, \]
\[ X_2 = t \partial_t + \left[ \left( \frac{1}{2} + \beta \right) u_n + \lambda_n \right] \partial u_n + \left[ \left( \frac{1}{2} - \beta \right) v_n + \mu_n \right] \partial v_n, \]
\[ X_3 = t^2 \partial_t + (tu_n + \rho_n) \partial u_n + (\gamma u_n + tv_n + \sigma_n) \partial v_n. \]

We use allowed transformations to set \( \alpha = 1, \xi_n = \eta_n = 0. \) The commutation relations then determine \( \beta = \frac{1}{2}, \gamma = -1. \) The functions \( \lambda_n(t), \mu_n(t), \rho_n(t) \) and \( \sigma_n(t) \) are greatly restricted by the commutation relations. As a matter of fact, we either have \( \lambda_n = \mu_n = \rho_n = \sigma_n = 0, \) or we can use allowed transformations to obtain \( \lambda_n = t, \mu_n = 1, \rho_n = 2t^2, \sigma_n = 2t. \)

We arrive at the following result.

**Theorem 1.** Precisely 4 classes of \( \mathfrak{sl}(2, \mathbb{R}) \) algebras can be realized by vector fields of the form (2.3). Any such \( \mathfrak{sl}(2, \mathbb{R}) \) algebra can be taken by an allowed transformation (2.6) into precisely one of the following algebras:
\[
\begin{align*}
\mathfrak{sl}(2, \mathbb{R})_1 : & \quad X_1 = v_n \partial_{u_n} \\
& \quad X_2 = \frac{1}{2}(u_n \partial_{u_n} - v_n \partial_{v_n}) \quad (4.7) \\
& \quad X_3 = u_n \partial_{v_n} \\
\mathfrak{sl}(2, \mathbb{R})_2 : & \quad X_1 = \partial_t \\
& \quad X_2 = t \partial_t + \frac{1}{2}(u_n \partial_{u_n} + v_n \partial_{v_n}) \quad (4.8) \\
& \quad X_3 = t^2 \partial_t + t(u_n \partial_{u_n} + v_n \partial_{v_n}) \\
\mathfrak{sl}(2, \mathbb{R})_3 : & \quad X_1 = \partial_t + v_n \partial_{u_n} \\
& \quad X_2 = t \partial_t + u_n \partial_{u_n} \quad (4.9) \\
& \quad X_3 = t^2 \partial_t + tu_n \partial_{u_n} + (tv_n - u_n) \partial_{v_n} \\
\mathfrak{sl}(2, \mathbb{R})_4 : & \quad X_1 = \partial_t + v_n \partial_{u_n} \\
& \quad X_2 = t \partial_t + (u_n + t) \partial_{u_n} + \partial_{v_n} \quad (4.10) \\
& \quad X_3 = t^2 \partial_t + (tv_n + 2t^2) \partial_{u_n} + (tv_n - u_n + 2t) \partial_{v_n}.
\end{align*}
\]

**IV.3 Indecomposable Lie algebras containing \( \mathfrak{sl}(2, \mathbb{R})_1 \)**

A Lie algebra \( L \) is called indecomposable if it cannot be written as a direct sum, \( L = L_1 \oplus L_2 \). A Lie algebra over \( \mathbb{R} \) containing \( \mathfrak{sl}(2, \mathbb{R}) \) is either simple or it allows a nontrivial Levi decomposition [15].
\[ L = S \triangleright R \quad (4.11) \]

where \( S \) is a semisimple Lie algebra and \( R \) is the radical, that is the maximal solvable ideal of \( L \).

It follows from the results of Section IV.1 that the only simple Lie algebras that can be constructed from operators of the form (2.3) are the 4 \( \mathfrak{sl}(2, \mathbb{R}) \) algebras obtained in section 4.2. We can hence concentrate on Lie algebras of the form (4.11).

The algebra \( S \) is either \( \mathfrak{sl}(2, \mathbb{R})_1 \) itself, or the direct sum of \( \mathfrak{sl}(2, \mathbb{R})_1 \) with one or more other \( \mathfrak{sl}(2, \mathbb{R}) \) algebras.

Requiring that a symmetry operator \( Y \) should commute with all elements of \( \mathfrak{sl}(2, \mathbb{R})_1 \) we find that \( Y \) must have the form

\[
Y_0 = \tau \partial_t + \left( \frac{1}{2} \hat{\tau} + \alpha \right) (u_n \partial_{u_n} + v_n \partial_{v_n}) . \quad (4.12)
\]

It is hence possible to construct precisely one semisimple Lie algebra properly containing \( \mathfrak{sl}(2, \mathbb{R})_1 \), namely the direct sum \( \mathfrak{sl}(2, \mathbb{R})_1 \oplus \mathfrak{sl}(2, \mathbb{R})_2 \) with \( \mathfrak{sl}(2, \mathbb{R})_2 \) defined in eq.(4.8).

Let us first introduce some notations for vector fields, to be used below. We put
\[ V(a_n) = a_n(u_n \partial u_n + v_n \partial v_n), \quad (4.13) \]
\[ T(a_n) = \partial_t + a_n(u_n \partial u_n + v_n \partial v_n), \quad (4.14) \]
\[ D(a_n) = t\partial_t + (1/2 + a_n)(u_n \partial u_n + v_n \partial v_n), \quad (4.15) \]
\[ P(a_n) = t^2\partial_t + (t + a_n)(u_n \partial u_n + v_n \partial v_n), \quad (4.16) \]
\[ R(a_n) = (t^2 + 1)\partial_t + (t + a_n)(u_n \partial u_n + v_n \partial v_n), \quad (4.17) \]
\[ Y_u(\lambda_n) = \lambda_n(t)\partial_{u_n}, \quad Y_v(\lambda_n) = \lambda_n(t)\partial_{v_n}. \quad (4.18) \]

In all cases we have \( \dot{a}_n = 0 \), but \( \lambda_n(t) \) can be a function of \( t \). Both \( a_n \) and \( \lambda_n(t) \) can be functions of \( n \).

Let us consider \( S = sl(2, \mathbb{R})_1 \) and \( S = sl(2, \mathbb{R})_1 \oplus sl(2, \mathbb{R})_2 \) in eq.(4.11) separately.

**A. \( S = sl(2, \mathbb{R})_1 \)**

The considered Lie algebras will have a basis \( \{X_1, X_2, X_3, Y_1, \ldots, Y_n\} \) with \( X_i \) given in eq.(4.7). The basis elements \( \{Y_1, \ldots, Y_n\} \) span the radical \( R \). The algebra \( S \) acts on \( R \) according to some linear, not necessarily irreducible, finite dimensional representation.

We start with the Cartan subalgebra \( \{X_2\} \) of \( sl(2, \mathbb{R}) \). It can be represented by a diagonal matrix in any finite dimensional representation. Consider \( Y \in R \). We have
with $Y$ as in eq.(2.3). Eq.(4.19) implies

$$p \tau = 0,$$

$$p \left( \frac{\dot{\tau}}{2} + a_n \right) = 0, \quad (p + \frac{1}{2}) \lambda_n = 0, \quad (p + 1) b_n = 0, \quad (4.20)$$

$$p \left( \frac{\dot{\tau}}{2} + d_n \right) = 0, \quad (p - \frac{1}{2}) \mu_n = 0, \quad (p - 1) c_n = 0.$$

For $p = 0$ we obtain an operator that commutes not only with $X_2$, but with all of $\text{sl}(2, \mathbb{R})_1$, namely $Y_0$ of eq.(4.12). This is a singlet representation of $\text{sl}(2, \mathbb{R})$.

For $p = 1$, or $p = -1$ eq.(4.19) forces $Y$ to be an element of $\text{sl}(2, \mathbb{R})_1$, in other words, no such $Y \in R$ exists.

For $p = \pm \frac{1}{2}$ we obtain $Y_1 = \lambda_n(t) \partial_{u_n}$ and $Y_2 = \mu_n(t) \partial_{v_n}$ respectively. Acting with $X_1$ and $X_3$ on these operators, we find that the only representation of $\text{sl}(2, \mathbb{R})_1$ that can be realized is a doublet one, namely $\{Y_u(\lambda_n), Y_v(\lambda_n)\}$ of eq.(4.18), with $\lambda_n(t)$ an arbitrary function of $n$ and $t$. The indecomposable Lie algebra $\{X_1, X_2, X_3, Y_u(\lambda_n), Y_v(\lambda_n)\}$ is isomorphic to the special affine Lie algebra $\text{saff}(2, \mathbb{R})$.

All further indecomposable symmetry algebras containing $\text{sl}(2, \mathbb{R})_1$ must be extensions of $\text{saff}(2, \mathbb{R})$. The objects that we can add to $\text{saff}(2, \mathbb{R})$ are either $\text{sl}(2, \mathbb{R})$ doublets or singlets. Let us run through all possibilities:
1. We can add an arbitrary number $k$ of doublets of the form (4.18) where the $k$ functions $\{\lambda^1_n(t), \lambda^2_n(t), \ldots, \lambda^k_n(t)\}$ must be linearly independent. However, we shall see in Section V that the presence of 3 such pairs forces the functions $F_n$ and $G_n$ in eq.(1.1) to be linear. Moreover, even two such pairs are compatible with a nonlinear interaction only if they are of the form (or transformable into):

$$\lambda^1_n(t) = 1, \quad \lambda^2_n(t) = t. \quad (4.21)$$

2. We can add a singlet of the form (4.12). If we have $\tau = 0$, then the commutation relations $[Y_0, Y_u]$ and $[Y_0, Y_v]$ imply $a_n = a_{n+1}$ and we can set $a_n = 1$. We obtain an affine Lie algebra gaff$(2, \mathbb{R})_1$ consisting of saff$(2, \mathbb{R})$ and $V(1)$ of eq.(4.13). If we have $\tau \neq 0$ in eq.(4.12) and only one operator of this type, then we can use allowed transformations to take $\tau(t)$ into $\tau(t) = 1$. The commutation relations $[Y_0, Y_u]$ and $[Y_0, Y_v]$ then imply

$$\lambda_n(t) = R_n e^{(a_n+k)t}, \quad \dot{R}_n = 0.$$

For $k = 0$ the algebra is decomposable. For $k \neq 0$ we can use allowed transformations to put $k = -1$ and $R_n = 1$. We obtain a second algebra isomorphic to gaff$(2, \mathbb{R})$, but not conjugate to the previous one. We have

$$\text{gaff}(2, \mathbb{R})_2 \sim \{X_1, X_2, X_3, Y_u(e^{(a_n-1)t}), Y_v(e^{(a_n-1)t}), T(a_n)\}. \quad (4.22)$$
In the special case of $a_n = a_{n+1}$ in eq. (4.22) a further extension is possible. We transform $\lambda = e^{(a-1)t}$ into $\lambda = 1$, then $T(a_n)$ goes into $D(b_n)$ with $b_n = b_{n+1} \equiv b \neq -\frac{1}{2}$, since for $b = -\frac{1}{2}$ the algebra is decomposable.

3. We can add 2 singlets of the form (4.12). If they commute, they must be $\{V(1), T(0)\}$. The obtained algebra is decomposable. If they do not commute, they must form a two dimensional Lie algebra, namely $\{T(0), D(a), a_n = a_{n+1} \equiv a\}$. This implies $\lambda_n(t) \sim 1$, i.e. the entire radical is $\{Y_u(1), Y_v(1), T(0), D(0)\}$ with $a \neq 1/2$ (the case $a = 1/2$ corresponds to a decomposable algebra).

4. If we add 3 singlets, the only case corresponds to the radical $\{Y_u(1), Y_v(1), V(1), T(0), D(0)\}$. There will then be no invariant interaction (see below).

5. Let us consider the special case of 2 doublets of the form (4.18), namely

$$Y_u(1) = \partial_{u_n}, \quad Y_v(1) = \partial_{v_n}, \quad Y_u(t) = t\partial_{u_n}, \quad Y_v(t) = t\partial_{v_n}. \quad (4.23)$$

This algebra can be extended by a further element, namely

$$Z = (\tau_0 + \tau_1 t + \tau_2 t^2) \partial_t + (\frac{4}{3} \tau_1 + \tau_2 t + a) (u_n \partial_{u_n} + v_n \partial_{v_n})$$

$$a_n = a_{n+1} \equiv a, \quad (4.24)$$

where $\tau_0$, $\tau_1$ and $\tau_2$ are constants. By allowed transformations we can
take $Z$ into one of the 4 operators $V(1)$, $T(a)$, $D(a)$ or $R(a)$ of (4.13), (4.14), (4.15) and (4.17), respectively.

6. We can add a two dimensional algebra to (4.23), namely

\[
\{T(0), D(a)\}, \ {T(0), V(1)}\}, \ \{V(1), D(0)\}, \ \text{or} \ \{V(1), R(0)\}.
\]

7. We can add only one three dimensional algebra to (4.23), namely

\[
\{T(0), D(0), V(1)\}.
\]

This completes the list of indecomposable symmetry algebras of the form (4.11) with $S = \mathfrak{sl}(2, \mathbb{R})_1$.

\section*{B. $S = \mathfrak{sl}(2, \mathbb{R})_1 \oplus \mathfrak{sl}(2, \mathbb{R})_2$}

The algebra $S$ is itself decomposable. It gives rise to precisely two indecomposable symmetry algebra. First we have the one obtained by adding the Abelian ideal (4.23) to $\mathfrak{sl}(2, \mathbb{R})_1 \oplus \mathfrak{sl}(2, \mathbb{R})_2$. Second, we get an eleven dimensional algebra by adding $V(1)$ to the first case.

\section*{IV.4 Decomposable Lie algebras containing $\mathfrak{sl}(2, \mathbb{R})_1$}

All decomposable Lie algebras $L_D$ can be obtained from the indecomposable $L_I$ ones, by adding their centralizers

\[
L_D = L_I \oplus C, \quad [C, L_I] = 0. \quad (4.25)
\]
The centralizer $C$ must commute with all elements of $\mathfrak{sl}(2, \mathbb{R})_1$ and hence all of its elements will have the form of $Y_0$ of eq.(4.12).

Let us consider the individual indecomposable algebras $L_I$.

1. $L_I = \mathfrak{sl}(2, \mathbb{R})_1$

The centralizer $C$ can be Abelian. Then we have the following possibilities: $C = \{ V(a_{i,n}) , \ i = 1, \ldots, k \}$ or $C = \{ V(a_{i,n}), T(b_n) , \ i = 1, \ldots, k \}$. The quantities $a_{1,n}, \ldots, a_{k,n}$ must form a set of $k$ linearly independent functions of $n$. If the centralizer is non-Abelian, then we have either $C \sim \mathfrak{sl}(2, \mathbb{R})_2$ or $C = \{ T(0), D(a) \}$. Both of these centralizers can be further extended by adding $V(a_{i,n}) , \ i = 1, \ldots, k$, (with $a_{1,n}, \ldots, a_{k,n}$ linearly independent).

2. $L_I = \mathfrak{saff}(2, \mathbb{R})$

We must require $Y_0$ of eq.(4.12) to commute with $Y_u(\lambda_n)$ and $Y_v(\lambda_n)$ of eq.(4.18). We obtain

$$\lambda_n \left( \frac{1}{2} \dot{\tau} + a_n \right) - \tau \dot{\lambda}_n = 0. \quad (4.26)$$

For $\tau = 0$ eq.(4.26) implies $\lambda_n a_n = 0$ and this is not allowed. For $\tau \neq 0$ we take $\tau \to 1$ by an allowed transformation, and eq.(4.26) then implies $\lambda_n(t) = \gamma_n e^{a_n t}$. A further allowed transformation will take
\( \gamma_n \to 1 \). We obtain the decomposable Lie algebra \( \text{saff}(2, \mathbb{R}) \oplus T(a_n) \).

In the special case \( a_n = a_{n+1} \) we transform \( \lambda_n(t) \to 1 \) and obtain a larger centralizer, namely \( \{ T(0), D(-\frac{1}{2}) \} \).

3. \( L_I = \text{gaff}(2, \mathbb{R})_1 \)

A nontrivial centralizer exists only if we have \( \lambda_n(t) = e^{a_n t} \) in \( \text{saff}(2, \mathbb{R}) \).

In the case \( a_n \neq 0 \), the centralizer is \( C = \{ T(a_n) \} \). If \( a_n = 0 \) the centralizer is \( C = \{ T(0), D(-\frac{1}{2}) \} \).

4. \( L_I = \text{gaff}(2, \mathbb{R})_2 \)

The centralizer is \( C = \{ T(a_n) - V(1) \} \). This algebra corresponds to the first one obtained in the case \( L_I = \text{gaff}(2, \mathbb{R})_1 \) above.

### IV.5 Summary of possible symmetry algebras containing \( sl(2, \mathbb{R})_1 \)

The classification of possible symmetry algebras can now be summed up rather simply. In addition to \( sl(2, \mathbb{R})_1 \) of eq.(4.7) we have a further algebra \( L_C \) (the “complementary” algebra). The structure of each symmetry algebra is

\[
L = sl(2, \mathbb{R})_1 + L_C, \quad [sl(2, \mathbb{R})_1, L_C] \subseteq L_C, \quad [L_C, L_C] \subseteq L_C. \quad (4.27)
\]
The symbol $\dot{+}$ denotes a direct sum of vector spaces. Moreover, eq.(4.27) shows that $L$ is either a direct sum or a semidirect one. The algebra $L_C$ is also a representation space for $sl(2, \mathbb{R})_1$. Irreducible representations in this case can be of dimension 1 or 2. All higher dimensional representations are completely reducible into sums of 1 and 2 dimensional representations.

For further use it is convenient to split the symmetry algebras into 4 series, according to the structure of the Lie algebra $L_C$.

**Series A**

$L_C$ is solvable and each element is an $sl(2, \mathbb{R})_1$ singlet. There exist 3 different infinite dimensional Lie algebras of this type:

\begin{align*}
A_1 & : \{V(a_{k,n})\} \\
A_2 & : \{T(b_n), V(a_{k,n})\} \\
A_3 & : \{T(0), D(b_n), V(a_{k,n})\}
\end{align*}

In each case we have $k = 1, 2, \ldots$ and the expressions $a_k$ must be linearly independent functions of $n$. Taking $1 \leq k \leq N$ for some finite $N$, we obtain finite dimensional subalgebras.

**Series B**

$L_C$ is solvable and contains precisely one $sl(2, \mathbb{R})_1$ doublet.

\[ B_1 = \{Y_u(\lambda_n), Y_v(\lambda_n)\} . \]
This is the indecomposable algebra $\text{saflf}(2, \mathbb{R})$ ($B_1$ together with $\text{sl}(2, \mathbb{R})_1$).

We have $\dim L = 5$.

$$B_2 = \{ Y_u(\lambda_n), Y_v(\lambda_n), V(1) \}.$$ (4.32)

$B_2$ is the indecomposable algebra $\text{gaff}(2, \mathbb{R})_1$ with $\dim L = 6$.

$$B_3 = \left\{ Y_u(e^{(a_n-1)t}), Y_v(e^{(a_n-1)t}), T(a_n) \right\}.$$ (4.33)

$B_3$ is the Lie algebra $\text{gaff}(2, \mathbb{R})_2$, isomorphic but not conjugate to $B_2$.

$$B_4 = \{ Y_u(e^{an}), Y_v(e^{an}), T(a_n) \}.$$ (4.34)

This algebra is $\text{saflf}(2, \mathbb{R}) \oplus T(a_n)$.

$$B_5 = \{ Y_u(1), Y_v(1), T(0), D(a) \}.$$ (4.35)

The algebra $B_5$ is indecomposable (except if $a = -1/2$).

$$B_6 = \left\{ Y_u(e^{(a_n-1)t}), Y_v(e^{(a_n-1)t}), T(a_n), V(1) \right\}.$$ (4.36)

The algebra $B_6$ is decomposable.

$$B_7 = \{ Y_u(1), Y_v(1), T(0), D(0), V(1) \}.$$ (4.37)

The algebra $B_7$ is indecomposable.

**Series C**

$L_C$ contains two $\text{sl}(2, \mathbb{R})$ doublets. The doublets could be characterized by any two functions $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$. However, we shall
only be interested in the case \( \lambda_1 = 1, \lambda_2 = t \). The others do not lead to invariant interactions. Similarly, we do not need algebras containing 3 or more doublets. In all cases the algebra \( L_C \) contains the elements (4.23). For \( \text{dim} \ L_C \geq 5 \) it contains further elements. We have

\[
C_1 = \{ Y_u(1), Y_e(1), Y_u(t), Y_e(t) \} . \tag{4.38}
\]

Further we just list the additional elements

\[
\begin{align*}
C_2. & \quad \{ T(a) \}, \ a = 0 \text{ or } 1 \tag{4.39} \\
C_3. & \quad \{ D(a) \} , \tag{4.40} \\
C_4. & \quad \{ R(a) \} , \tag{4.41} \\
C_5. & \quad \{ V(1) \} , \tag{4.42} \\
C_6. & \quad \{ T(0), D(a) \} , \tag{4.43}
\end{align*}
\]

In all cases above, \( a \) does not depend on \( n \ (a_{n+1} = a_n) \).

\[
\begin{align*}
C_7. & \quad \{ V(1), T(0) \} , \tag{4.44} \\
C_8. & \quad \{ V(1), D(0) \} , \tag{4.45} \\
C_9. & \quad \{ V(1), R(0) \} , \tag{4.46} \\
C_{10}. & \quad \{ T(0), D(0), P(0) \} \sim \mathfrak{sl}(2, \mathbb{R})_2 . \tag{4.47} \\
C_{11}. & \quad \{ T(0), D(0), V(1) \} \tag{4.48} \\
C_{12}. & \quad \{ T(0), D(0), P(0), V(1) \} \tag{4.49}
\end{align*}
\]

**Series D**

\[28\]
$L_C$ contains $sl(2, \mathbb{R})_2$ and (possibly) further elements, namely

1. None, \hspace{1cm} (4.50)
2. $\{V(a_n)\}$, \hspace{1cm} (4.51)
3. $\{V(a_{1,n}), V(a_{2,n})\}$, \hspace{1cm} (4.52)
4. $\{Y_u(1), Y_v(1), Y_u(t), Y_v(t)\}$ \hspace{1cm} (4.53)
5. $\{Y_u(1), Y_v(1), Y_u(t), Y_v(t), V(1)\}$ \hspace{1cm} (4.54)

($D_4$ coincides with $C_{10}$ and $D_5$ with $C_{12}$).

**V THE INVARIANT INTERACTIONS**

**V.1 General Procedure and Interactions Invariant under $SL(2, \mathbb{R})_1$**

In this section we shall find all interaction functions, invariant under symmetry groups, containing $SL(2, \mathbb{R})_1$. We make use of the subalgebra classification provided in Section IV.

We first establish the form of the interaction, invariant under $SL(2, \mathbb{R})_1$ itself. To do this we set $\tau(t) = \lambda_n(t) = \mu_n(t) = 0$ in the determining equations (2.4) and (2.5) and consider the equations obtained for $a_n = -d_n = 1$, $b_n = c_n = 0$, then $b_n = 1$, $a_n = -d_n = c_n = 0$, and finally $c_n = 1$, $a_n = -d_n = b_n = 0$. The general solution of the obtained system of 6 equations can be written...
in the following form:

\[ F_n = u_{n+1} f_n + u_n g_n, \]
\[ G_n = v_{n+1} f_n + v_n g_n, \]

where \( f_n \) and \( g_n \) are functions of 4 variables each, namely

\[ t, \; \xi_n = u_{n+1} v_{n-1} - u_{n-1} v_{n+1}, \]
\[ \xi_\alpha = u_\alpha v_n - u_n v_\alpha, \; \alpha = n \pm 1. \]

(5.2)

Note that \( \xi_n, \xi_{n+1} \) and \( \xi_{n-1} \) are as given in eq.(5.2). They are not upshifts or downshifts of each other.

We shall proceed further by dimension of the symmetry algebra and by its structure. Thus we can successively add \( \mathfrak{sl}(2,\mathbb{R}) \) singlets of the form (4.12) or doublets of the form (4.18). We continue adding symmetry elements, until the interaction is completely specified, i.e. involves no further arbitrary functions. We then solve the “inverse problem”. That is, we substitute the functions \( F_n \) and \( G_n \) back into the determining equations and solve for the symmetries. This provides a verification of previous calculations. More important, this procedure will find the largest symmetry algebra, allowed by any given interaction.

Obviously, all invariant interactions will have the form (5.1). It is the functions \( f_n \) and \( g_n \) that will be further refined and their dependence on the variables \( \xi_k \) and \( t \) will be restricted.

For future convenience we write down two further forms of the \( \text{SL}(2,\mathbb{R})_1 \) invariant interaction functions, equivalent to (5.1). The first
is

\[ F_n = u_{n+1} \frac{\xi_{n-1}}{\xi_n} h_n + u_n k_n, \] (5.3)

\[ G_n = v_{n+1} \frac{\xi_{n-1}}{\xi_n} h_n + v_n k_n, \]

where \( h_n \) and \( k_n \) are arbitrary functions of the variables (5.2). The second convenient form is

\[ F_n = (\lambda_{n-1} u_{n+1} - \lambda_{n+1} u_{n-1}) \phi_n + (\lambda_{n+1} u_n - \lambda_n u_{n+1}) \psi_n + \frac{\lambda_n}{\lambda_{n+1}} u_{n+1}, \] (5.4)

\[ G_n = (\lambda_{n-1} v_{n+1} - \lambda_{n+1} v_{n-1}) \phi_n + (\lambda_{n+1} v_n - \lambda_n v_{n+1}) \psi_n + \frac{\lambda_n}{\lambda_{n+1}} v_{n+1}, \]

where \( \lambda_n(t) \) is some arbitrary function of \( n \) and \( t \) and \( \phi_n \) and \( \psi_n \) depend in an unspecified manner on the variables (5.2).

**V.2 Interactions Invariant under Four dimensional Symmetry Groups**

As was shown in Section IV, two types of 4 dimensional symmetry algebras containing \( \mathfrak{sl}(2, \mathbb{R})_1 \) can exist. Both are decomposable according to the pattern \( 4 = 3 + 1 \). Here and below we shall always list the operators that we can add to \( \mathfrak{sl}(2, \mathbb{R})_1 \).

1. \( V(a_n) = a_n (u_n \partial u_n + v_n \partial v_n) \) (5.5)

The invariant interactions will have the form (5.3), but \( h_n \) and \( k_n \) will depend on 3 variables only.
(i) $a_{n-1} + a_{n+1} \neq 0$.

The variables are

$$t, \quad \eta_\alpha = (\xi_\alpha)^{a_{n-1}+a_{n+1}}(\xi_n)^{-a_n-a_\alpha}, \quad \alpha = n \pm 1.$$  \hfill (5.6)

(ii) $a_{n-1} + a_{n+1} = 0$.

The variables are

$$t, \quad \xi_n, \quad \eta = (\xi_{n+1})^{a_{n+1}-a_n}(\xi_{n-1})^{a_n+a_{n+1}}.$$  \hfill (5.7)

2. $T(b_n) = \partial_t + b_n(u_n\partial_{u_n} + v_n\partial_{v_n})$.  \hfill (5.8)

The invariant interaction will again have the form (5.3), however in this case $h_n$ and $k_n$ are arbitrary functions of the 3 variables

$$\zeta_n = \xi_n e^{-(b_{n-1}+b_{n+1})t},$$

$$\zeta_\alpha = \xi_\alpha e^{-(b_n+b_\alpha)t}, \quad \alpha = n \pm 1.$$  \hfill (5.9)

We see that adding further singlets of the type $V(a_n)$ will restrict the variables in the functions $h_n$ and $k_n$, not however the general form of eq.(5.3).

### V.3 Five dimensional Symmetry Groups

From the results of Section IV we know that 3 decomposable and 1 indecomposable symmetry algebras of dimension 5 can exist. Let us run through all four possibilities.

**Decomposition** $5 = 3 + 1 + 1$
1. \( V(a_{i,n}) = a_{i,n}(u_n \partial u_n + v_n \partial v_n) \), \( i = 1, 2, \quad a_{2,n} \neq \lambda a_{1,n}. \) \hfill (5.10)

The interaction is of the form (5.3). The functions \( h_n \) and \( k_n \) depend on 2 variables each, namely time \( t \) and

\[
\eta = (\xi_{n-1})^A(\xi_{n+1})^B(\xi_n)^C, \quad (5.11)
\]

\[
A = a_{1,n}(a_{2,n+1} + a_{2,n-1}) + a_{1,n+1}(a_{2,n-1} - a_{2,n}) - a_{1,n-1}(a_{2,n+1} + a_{2,n}),
\]

\[
B = -a_{1,n}(a_{2,n+1} + a_{2,n-1}) + a_{1,n+1}(a_{2,n-1} + a_{2,n}) - a_{1,n-1}(a_{2,n+1} - a_{2,n}),
\]

\[
C = a_{1,n}(a_{2,n+1} - a_{2,n-1}) - a_{1,n+1}(a_{2,n-1} + a_{2,n}) + a_{1,n-1}(a_{2,n+1} + a_{2,n}). \quad (5.12)
\]

Note that the variable \( \eta \) always exists since the condition \( A = B = C = 0 \) (and hence \( \eta = \text{const} \)) only occurs for \( a_{1,n-1}a_{2,n} - a_{1,n}a_{2,n-1} = 0 \), which implies \( a_{2,n} = \lambda a_{1,n}, \lambda = \text{const}, \) and this is not allowed.

2. \( V(a_n) = a_n(u_n \partial u_n + v_n \partial v_n), \quad T(b_n) = \partial_t + b_n(u_n \partial u_n + v_n \partial v_n). \)

The invariant interaction is as in eq.(5.3) with \( h_n \) and \( k_n \) functions of 2 variables each. Namely:

(i) \( a_{n+1} + a_{n-1} \neq 0 \)

\[
\rho_\alpha = (\zeta_\alpha)^{a_{n+1} + a_{n-1}}(\zeta_n)^{-a_n - a_n}, \quad \alpha = n \pm 1 \quad (5.13)
\]

with \( \zeta_\alpha, \zeta_n \) as in eq.(5.9).

(ii) \( a_{n+1} + a_{n-1} = 0 \)

\[
\rho_n = \zeta_n, \quad \sigma_n = (\zeta_{n-1})^{a_{n+1} + a_n}(\zeta_{n+1})^{a_{n+1} - a_n}. \quad (5.14)
\]
Decomposition $5 = 3 + 2$

3. $T(0) = \partial_t, \quad D(b_n) = t\partial_t + \left(\frac{1}{2} + b_n\right)(u_n\partial_{u_n} + v_n\partial_{v_n}) \quad (5.15)$

We impose $b_n \neq -\frac{1}{2}$, otherwise we have no invariant interaction.

We must distinguish 2 subcases here.

(i) $b_{n+1} + b_{n-1} + 1 \neq 0$

Interaction as in eq.(5.3) with

$$h_n = (\xi_n)^{-\frac{2}{b_{n+1} + b_{n-1} + 1}} p_n, \quad k_n = (\xi_n)^{-\frac{2}{b_{n+1} + b_{n-1} + 1}} q_n, \quad (5.16)$$

where $p_n$ and $q_n$ depend on 2 variables, namely

$$\chi_\alpha = (\xi_\alpha)^{b_{n+1} + b_{n-1} + 1}(\xi_n)^{-b_n - b_\alpha - 1}, \quad \alpha = n \pm 1. \quad (5.17)$$

(ii) $b_{n+1} + b_{n-1} + 1 = 0, \quad b_{n+1} + b_n + 1 \neq 0$

$$h_n = (\xi_{n+1})^{-\frac{2}{b_{n+1} + b_n + 1}} p_n, \quad k_n = (\xi_{n+1})^{-\frac{2}{b_{n+1} + b_n + 1}} q_n, \quad (5.18)$$

where $p_n$ and $q_n$ depend on

$$\chi_n = (\xi_{n-1})^{b_{n+1} + b_n + 1}(\xi_{n+1})^{-b_{n-1} - b_n - 1}, \quad \xi_n. \quad (5.19)$$

Note that for $b_{n+1} + b_{n-1} + 1 = 0, \quad b_{n+1} + b_n + 1 = 0$ we have $b_n = -1/2$ and there is no invariant interaction.
Indecomposable Lie algebra

4. $Y_u(\lambda_n) = \lambda_n(t) \partial_u$, $Y_v(\lambda_n) = \lambda_n(t) \partial_v$. \hspace{1cm} (5.20)

The invariant interaction is as in eq.(5.4), but the functions $\phi_n$ and $\psi_n$ depend on only 2 variables, namely

$$t, \ \omega = \lambda_{n-1}\xi_{n+1} - \lambda_n\xi_n - \lambda_{n+1}\xi_{n-1}. \hspace{1cm} (5.21)$$

V.4 Six dimensional Symmetry Groups

Decomposition $6 = 3 + 1 + 1 + 1$

1. $V(a_{i,n}) = a_{i,n}(u_n \partial_u + v_n \partial_v), \ i = 1, 2, 3. \hspace{1cm} (5.22)$

The invariant interaction is as in eq.(5.3) but $h_n$ and $k_n$ are functions of $t$ only. We see that the coefficients $a_{i,n}$ do not figure in the interaction. Hence, we can add an arbitrary number of vector fields $V(a_{i,n}), i \in \mathbb{Z}$ to the symmetry algebra. In other words, the symmetry algebra for the interaction (5.3) with $h_n$ and $k_n$ depending on $t$ alone is infinite dimensional.

2. $V(a_{i,n}) = a_{i,n}(u_n \partial_u + v_n \partial_v), \ i = 1, 2,$

$T(b_n) = \partial_t + b_n(u_n \partial_u + v_n \partial_v).$

The invariant interaction is as in eq.(5.3) but $h_n$ and $k_n$ depend on 1 variable only, namely

$$\omega = \eta e^{-2|\mathbf{M}|}, \ M = \begin{pmatrix} b_{n-1} & b_n & b_{n+1} \\ a_{1,n-1} & a_{1,n} & a_{1,n+1} \\ a_{2,n-1} & a_{2,n} & a_{2,n+1} \end{pmatrix} \hspace{1cm} (5.23)$$
with  \( \eta \) as in eq.(5.11).

**Decomposition 6 = 3 + 2 + 1**

3. \( V(a_n) = a_n(u_n \partial u_n + v_n \partial v_n), \quad T(0) = \partial_t \),

\( D(c_n) = t \partial t + \left( \frac{1}{2} + c_n \right)(u_n \partial u_n + v_n \partial v_n). \)

We start from eq.(5.3). The presence of \( T(0) = \partial_t \) implies that \( h_n \) and \( k_n \) do not depend on \( t \). We first notice that if we have

\[
\gamma_n = c_n + \frac{1}{2} = 0 \quad \text{or} \quad \gamma_n = c_n + \frac{1}{2} = \lambda a_n \quad (5.24)
\]

then we must have \( h_n = k_n = 0 \) (no invariant interaction). In all other cases, invariance under \( V(a_n) \) and \( D(c_n) \) implies

\[
h_n = (\xi_n)^\mu(\xi_{n+1})^\nu(\xi_{n-1})^\rho p_n(\omega), \quad k_n = (\xi_n)^\mu(\xi_{n+1})^\nu(\xi_{n-1})^\rho q_n(\omega),
\]

\[
\omega = (\xi_{n-1})^A(\xi_{n+1})^B(\xi_n)^C \quad (5.25)
\]

with \( A, B \) and \( C \) as in eq.(5.12) with the substitutions

\[
a_{1,n} \rightarrow c_n + \frac{1}{2}, \quad a_{2,n} \rightarrow a_n.
\]

The constants \( \mu, \nu \) and \( \rho \) in eq.(5.25) satisfy

\[
(a_{n+1} + a_{n-1})\mu + (a_{n+1} + a_n)\nu + (a_{n-1} + a_n)\rho = 0, \quad \gamma_{n+1} + \gamma_{n-1})\mu + (\gamma_{n+1} + \gamma_n)\nu + (\gamma_{n-1} + \gamma_n)\rho = -2. \quad (5.26)
\]

Thus, for \( C \neq 0 \) we can put

\[
\mu = 0, \quad \nu = 2 \frac{a_n + a_{n-1}}{C}, \quad \rho = -2 \frac{a_n + a_{n+1}}{C}.
\]
For $C = 0$, $A \neq 0$

$$\mu = 2 \frac{a_n + a_{n+1}}{A}, \quad \nu = -2 \frac{a_{n+1} + a_{n-1}}{A}, \quad \rho = 0.$$  

For $C = A = 0$, $B \neq 0$

$$\mu = -2 \frac{a_{n-1} + a_n}{B}, \quad \nu = 0, \quad \rho = 2 \frac{a_{n+1} + a_{n-1}}{B}.$$  

The case $A = B = C = 0$ corresponds to eq.(5.24) and hence to the absence of an invariant interaction.

Decomposition 6 = 3 + 3

4. $sl(2, \mathbb{R})_1 \oplus sl(2, \mathbb{R})_2$

The algebra $sl(2, \mathbb{R})_2$ is as in eq.(4.8) and the invariant interaction is

$$F_n = \frac{1}{(\xi_n)^2} \left[ u_{n+1} \frac{\xi_{n-1}}{\xi_n} p_n(\chi_{n+1}, \chi_{n-1}) + u_n q_n(\chi_{n+1}, \chi_{n-1}) \right],$$

$$G_n = \frac{1}{(\xi_n)^2} \left[ v_{n+1} \frac{\xi_{n-1}}{\xi_n} p_n(\chi_{n+1}, \chi_{n-1}) + v_n q_n(\chi_{n+1}, \chi_{n-1}) \right],$$

$$\chi_{n+1} = \frac{\xi_{n+1}}{\xi_n}, \quad \chi_{n-1} = \frac{\xi_{n-1}}{\xi_n}.$$  

Decomposition 6 = 5 + 1

5. $saff(2) \oplus A_1$

We have

$$Y_u(e^{a_n t}) = e^{a_n t} \partial_{u_n}, \quad Y_v(e^{a_n t}) = e^{a_n t} \partial_{v_n}, \quad T(a_n) = \partial_t + a_n(u_n \partial_{u_n} + v_n \partial_{v_n}).$$
The invariant interaction will be as in eq.(5.4) with $\lambda_n = e^{a_n t}$. The functions $\phi_n$ and $\psi_n$ will satisfy

$$
\phi_n = e^{(a_n-a_{n-1}-a_{n+1})t}K_n(\omega), \quad \psi_n = e^{-a_{n+1}t}L_n(\omega),
$$

$$
\omega = e^{-(a_n+a_{n+1})t}\xi_{n+1} - e^{-(a_{n+1}+a_{n-1})t}\xi_n - e^{-(a_{n-1}+a_n) t}\xi_{n-1}.
$$

(5.30)

**Indecomposable symmetry algebras**

It was shown in Section IV that two inequivalent gaff(2) symmetry algebras exist.

6. gaff(2, $\mathbb{R}$)$_1$

$$
Y_u(\lambda_n) = \lambda_n(t)\partial_u, \quad Y_v(\lambda_n) = \lambda_n(t)\partial_v, \quad V(1) = u_n\partial_u + v_n\partial_v.
$$

The interaction is as in eq.(5.4), however $\phi_n$ and $\psi_n$ depend only on $t$. This means that the equations are linear and moreover the equations (1.1) for $u_n$ and $v_n$ are decoupled.

7. gaff(2, $\mathbb{R}$)$_2$

The algebra is as in eq.(4.22) (or (4.33)), the interaction as in eq.(5.4) with $\lambda_n(t) = e^{(a_n-1)t}$. The functions $\phi_n$ and $\psi_n$ satisfy

$$
\phi_n = e^{-(a_{n+1}+a_{n-1})t}K_n(\omega),
$$

$$
\psi_n = e^{(-a_{n+1})t}L_n(\omega),
$$

$\omega$ as in eq.(5.30).

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V.5 Seven dimensional Symmetry Groups

**Decomposition 7 = 3 + 1 + 1 + 1 + 1**

We exclude the case

\[ V(a_{i,n}) = a_{i,n}(u_n \partial_{u_n} + v_n \partial_{v_n}), \quad i = 1, \ldots, 4 \]

since the only invariant interaction is (5.3) with \( h_n \) and \( k_n \) functions of \( t \). We already know that the symmetry algebra is infinite dimensional.

1. \( V(a_{i,n}) = a_{i,n}(u_n \partial_{u_n} + v_n \partial_{v_n}), \quad i = 1, 2, 3, \)
   \[ T(b_n) = \partial_t + b_n(u_n \partial_{u_n} + v_n \partial_{v_n}). \]

   The interaction is as in eq.(5.3) with \( h_n \) and \( k_n \) constants (depending on \( n \)). The algebra is actually infinite dimensional: we can take any number of operators \( V(a_{i,n}) \).

**Decomposition 7 = 3 + 2 + 1 + 1**

2. \( V(a_{i,n}) = a_{i,n}(u_n \partial_{u_n} + v_n \partial_{v_n}), \quad i = 1, 2, \)
   \[ T(0) = \partial_t, \quad D(c_n) = t\partial_t + (\frac{1}{2} + c_n)(u_n \partial_{u_n} + v_n \partial_{v_n}). \]

   We put \( \gamma_n = c_n + \frac{1}{2} \). An invariant interaction exists if and only if we have

\[
\Delta = \det \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \gamma_{n-1} \\
 a_{1,n} & a_{1,n+1} & a_{1,n-1} \\
 a_{2,n} & a_{2,n+1} & a_{2,n-1}
\end{pmatrix} \neq 0. \tag{5.34}
\]

The invariant interaction is that of eq.(5.3) with

\[
h_n = \eta^k p_n, \quad k_n = \eta^k q_n, \quad k = -\frac{2}{\Delta}. \tag{5.35}
\]

The variable \( \eta \) is as in eq.(5.11), \( p_n \) and \( q_n \) are constants.
Decomposition 7 = 3 + 3 + 1

3. $\text{sl}(2, \mathbb{R})_1 \oplus \text{sl}(2, \mathbb{R})_2 \oplus A_1$

We have $A_1 = \{V(a_n)\}$. The invariant interaction can be obtained from eq.(5.28). The additional invariance implied by the presence of $V(a_n)$ restricts $p_n$ and $q_n$ to

$$
p_n = \left( \frac{\xi_{n+1}}{\xi_n} \right) \frac{\gamma^{a_{n+1}+a_{n-1}}}{a_n-a_{n-1}} r_n(\omega),
$$

$$
q_n = \left( \frac{\xi_{n+1}}{\xi_n} \right) \frac{\gamma^{a_{n+1}+a_{n-1}}}{a_n-a_{n-1}} s_n(\omega),
$$

$$
\omega = (\xi_{n+1})^{a_{n+1}-a_n} (\xi_{n-1})^{a_n-a_{n-1}} (\xi_n)^{a_{n-1}-a_{n+1}}
$$

and we must impose $a_n \neq a_{n-1}$ (otherwise we have $F_n = G_n = 0$).

Decomposition 7 = 6 + 1

The algebra gaff(2, $\mathbb{R}$)$_1$ does not allow any nonlinear interactions. Let us consider gaff(2, $\mathbb{R}$)$_2$ of eq.(4.22).

4. $\text{gaff}(2, \mathbb{R})_2 \oplus \{U = u_n \partial u_n + v_n \partial v_n\}$

The interaction is as in eq.(5.4) with $\phi_n$ and $\psi_n$ as in eq.(5.33). Invariance under the dilations corresponding to $U$ implies that $\phi_n$ and $\psi_n$ do not depend on $\omega$. Hence the interaction is linear and decoupled.

Indecomposable Lie algebras
5. \( Y_u(\lambda_n) = \lambda_n(t) \partial_{u_n}, \quad Y_v(\lambda_n) = \lambda_n(t) \partial_{v_n}, \)
\( Y_u(\mu_n) = \mu_n(t) \partial_{u_n}, \quad Y_v(\mu_n) = \mu_n(t) \partial_{v_n}. \)

We start from eq.(5.4) with \( \phi_n \) and \( \psi_n \) functions of \( t \) and \( \omega \) as
in eq.(5.21). If \( \phi_n \) and \( \psi_n \) do not depend on \( \omega \), the interaction is
already linear and decoupled. Hence, \( \omega \) must be invariant under the
transformations corresponding to \( Y_u(\mu_n) \) and \( Y_v(\mu_n) \). This implies
that \( \lambda_n \) and \( \mu_n \) are independent of \( n \). Further, invariance implies
\[
\frac{\ddot{\lambda}_n}{\lambda_n} = \frac{\ddot{\mu}_n}{\mu_n} = \tilde{k}
\]  
(5.39)

with \( \tilde{k} = \text{const} \). Eq.(5.39) allows solutions
\[
\begin{pmatrix}
\lambda_n \\
\mu_n
\end{pmatrix} = \begin{pmatrix}
sin kt \\
\cosh kt
\end{pmatrix}, \quad \begin{pmatrix}
sinh kt \\
\cos kt
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
t
\end{pmatrix}.
\]  
(5.40)

These solutions are all equivalent under allowed transformations. We
choose \( \lambda_n = 1, \mu_n = t \), i.e.
\[
Y_u(1) = \partial_{u_n}, \quad Y_v(1) = \partial_{v_n}, \quad Y_u(t) = t \partial_{u_n}, \quad Y_v(t) = t \partial_{v_n},
\]  
(5.41)

The invariant interaction is
\[
F_n = (u_{n+1} - u_{n-1}) \phi_n(\omega, t) + (u_n - u_{n+1}) \psi_n(\omega, t),
\]
\[
G_n = (v_{n+1} - v_{n-1}) \phi_n(\omega, t) + (v_n - v_{n+1}) \psi_n(\omega, t)
\]  
(5.42)

with
\[
\omega = \xi_{n+1} - \xi_{n-1} - \xi_n.
\]  
(5.43)

6. \( Y_u(1) = \partial_{u_n}, \quad Y_v(1) = \partial_{v_n}, \quad T(0) = \partial_t, \)
\( D(b) = t \partial_t + \left( \frac{1}{2} + b \right) (u_n \partial_{u_n} + v_n \partial_{v_n}), \quad b \neq -\frac{1}{2}, \)
\( b = \text{const}. \)
The invariant interaction is as in eq.(5.42) with

$$
\phi_n = k_n \omega^{-\frac{2}{m+1}}, \quad \psi_n = p_n \omega^{-\frac{2}{m+1}}
$$

(5.44)

with \(k_n\) and \(p_n\) constants, \(\omega\) as in eq.(5.43). For \(b = -\frac{1}{2}\) there is no invariant interaction. For \(b \neq -\frac{1}{2}\) the symmetry algebra is actually larger and includes \(Y_u(t) = t \partial u_n\) and \(Y_v(t) = t \partial v_n\).

### V.6 Symmetry Groups of Dimensions 8, 9 and 10

By now all invariant interactions have been specified up to arbitrary constants (depending on \(n\)), except those involving symmetry algebras containing the subalgebra \(sl(2, \mathbb{R})_1 \oplus sl(2, \mathbb{R})_2\), or the subalgebra \(\{Y_u(1), Y_v(1), Y_u(t), Y_v(t)\}\) of eq.(5.41). Let us consider the remaining nonlinear interactions.

1. \(sl(2, \mathbb{R})_1 \oplus sl(2, \mathbb{R})_2 \oplus \{V(a_{1,n})\} \oplus \{V(a_{2,n})\}\) (5.45)

The invariant interaction is obtained from eq.(5.37) by specifying \(r_n(\omega)\) and \(s_n(\omega)\) to be specific powers of \(\omega\). The result is

$$
F_n = \xi_n^{-2} \left[ u_{n+1} + \frac{\xi_{n-1}}{\xi_n} p_n + u_n q_n \right] (\xi_{n-1})^{-\frac{2}{m}} (\xi_{n+1})^{-\frac{2}{m}} (\xi_n)^{\frac{4+2m}{m}},
$$

$$
G_n = \xi_n^{-2} \left[ v_{n+1} + \frac{\xi_{n-1}}{\xi_n} p_n + v_n q_n \right] (\xi_{n-1})^{-\frac{2}{m}} (\xi_{n+1})^{-\frac{2}{m}} (\xi_n)^{\frac{4+2m}{m}}.
$$

(5.46)

Here \(p_n\) and \(q_n\) are constants, \(A\) and \(B\) are as in eq.(5.12) and

$$
D = a_{1,n} (a_{2,n+1} - a_{2,n-1}) + a_{1,n+1} (a_{2,n-1} - a_{2,n}) + a_{1,n-1} (a_{2,n} - a_{2,n+1}).
$$

(5.47)
We assume \( D \neq 0 \), otherwise there is no invariant interaction. In particular, we have \( a_{1,n} \neq a_{1,n+1}, a_{2,n} \neq a_{2,n+1} \).

2. Algebras containing \( (Y_u(1), Y_v(1), Y_u(t), Y_v(t)) \) of (5.41) plus one additional operator \( Z \).

The interaction is as in eq.(5.42) with a restriction on \( \phi_n \) and \( \psi_n \).

(i) \( Z = T(a) = \partial_t + a(u_n \partial_{u_n} + v_n \partial_{v_n}), \quad a \equiv a_n = a_{n+1} \),

\[
\phi_n = \phi_n(\eta), \quad \psi_n = \psi_n(\eta), \quad \eta = \omega e^{-2\alpha t}. \quad (5.48)
\]

(ii) \( Z = D(a) = t \partial_t + (\frac{1}{2} + a)(u_n \partial_{u_n} + v_n \partial_{v_n}), \quad a \equiv a_n = a_{n+1} \),

\[
\phi_n = \frac{1}{t^2} r_n(\eta), \quad \psi_n = \frac{1}{t^2} s_n(\eta), \quad \eta = \omega t^{-(2a+1)}. \quad (5.49)
\]

(iii) \( Z = R(b) = (t^2 + 1) \partial_t + (t + b)(u_n \partial_{u_n} + v_n \partial_{v_n}), \quad b \equiv b_n = b_{n+1} \),

\[
\phi_n = \frac{1}{(t^2 + 1)^2} r_n(\eta), \quad \psi_n = \frac{1}{(t^2 + 1)^2} s_n(\eta), \quad \eta = \frac{\omega}{1 + t^2} e^{-2b \arctg t} \quad (5.50)
\]

with \( \omega \) as in eq.(5.43) in all cases.

(iv) \( Z = V(1) \). Then \( \phi_n \) and \( \psi_n \) depend only on \( t \) and the interaction is linear.

We can add 2 operators to those of eq.(5.41)

\[
T(0) = \partial_t, \quad D(b) = t \partial_t + (\frac{1}{2} + b)(u_n \partial_{u_n} + v_n \partial_{v_n}).
\]

The invariant interaction coincides with that of eq.(5.44).
Finally, the interaction (5.42) is invariant under a 10 dimensional symmetry algebra of the form

\[(\mathfrak{sl}(2, \mathbb{R})_1 \oplus \mathfrak{sl}(2, \mathbb{R})_2) \triangleright \{Y_u(1), Y_v(1), Y_u(t), Y_v(t)\}\]

for

\[
\phi_n = k_n \omega^{-2}, \quad \psi_n = p_n \omega^{-2},
\]

i.e. \(b = 0\) in eq.(5.44).

**VI  Summary and Conclusions**

Let us first sum up the results on invariant interactions and the corresponding symmetry algebras. We shall follow the summary of possible symmetry algebras outlined in Section IV.5. The results are presented in tables.

**Table 1.** The Series A of symmetry algebras. The algebra \(L_C\) of eq.(4.27) consists entirely of \(\mathfrak{sl}(2, \mathbb{R})_1\) singlets. In the first column of Table 1 we list the symmetry algebras. The number in brackets (e.g. \(A_1(3)\)) denotes the dimension of the symmetry algebra. The notation for basis elements in column 2 are as in eq.(4.13)-(4.18). Note that if the functions \(h_n\) and \(k_n\) in the interaction (5.3) depend only on \(t\) or are constants, then the symmetry algebra is infinite dimensional, although the interaction is nonlinear.

The case \(A_3(7)\) corresponds to an algebra \(L\) with \(\text{dim } L = 7\) and the interaction is completely specified. (see (5.3), (5.34)-(5.35) ).
other cases the functions $h_n$ and $k_n$ depend on 1, 2 or 3 variables involving $u_k$ and $v_k$.

**Table 2.** The Series B of symmetry algebras. The symmetry algebras are either 5 or 6 dimensional. The interactions are as in eq.(5.4) and involve two arbitrary functions $\phi_n$ and $\psi_n$. A B-type symmetry allows $\phi_n$ and $\psi_n$ to depend on just one variable involving $u_k$ and $v_k$. Any extension of the B type algebras will restrict $\lambda_n(t)$ to be $\lambda_n = 1$ and will involve a further pair with $\lambda_n = t$. This takes us into the series C of symmetry algebras.

The algebras $B_2$, $B_6$ and $B_7$ of eq.(4.32), (4.36) and (4.37) lead to linear interactions. Any interaction invariant with respect to $B_5$ will be invariant under a larger group, corresponding to a Lie algebra in the series C. We do not include linear interactions in the tables and we list interactions together with their\textit{ maximal} symmetry algebras.

**Table 3.** The Series C of symmetry algebras. The interaction will be as in eq.(5.42) involving a variable $\omega$ as in eq.(5.43). The algebras $C_5(8)$, $C_7(9)$, $C_8(9)$, $C_9(9)$, $C_{11}(10)$, $C_{12}(11)$ absent in the table, lead to a linear interaction.

For $C_6(9)$ and $C_{10}(10)$ the interactions are specified up to constants (that can depend on $n$). In all other cases, the arbitrary functions depend on one variable, involving $u_k$ and $v_k$.

**Table 4.** The Series D of symmetry algebras. There are 3 such algebras of dimension 6, 7 and 8, respectively. They all lead to nontrivial
invariant interactions of the form (5.28). For $D_3(8)$ the interaction is completely specified. We do not list $D_4(10)$ in Table 4 since it coincides with $C_{10}(10)$ of Table 3. The algebra $D_5(11)$ corresponds to a linear interaction.

For each interaction we have verified that the given symmetry algebra is the maximal one.

A few words about the interpretation of the invariant interactions. From eq.(5.1) and the variables (5.2) we see that invariance under $sl(2, \mathbb{R})_1$ imposes very strong restrictions.

1. In particular, if the interaction is linear and $sl(2, \mathbb{R})_1$ invariant, we must have

$$F_n = \sum_{k=n-1}^{n+1} A_k(t)u_k, \quad G_n = \sum_{k=n-1}^{n+1} A_k(t)v_k, \quad (6.1)$$

i.e. the equations (1.1) for $u_k$ and $v_k$ decouple (into identical equations for $u_n$ and $v_n$ separately).

2. If the interaction terms $F_n$ and $G_n$ in eq.(5.1) are nonlinear, they always involve many-body forces. That is, they cannot be written as sums of terms of the type $h_n(u_n, v_n)$ or $h_n(u_n, v_{n+1})$, etc... Indeed, each invariant variable $\xi_n, \xi_{n+1}, \xi_{n-1}$ itself involves 4 of the original variables $u_i, v_i$ simultaneously. This many-body character becomes more pronounced when the invariance algebra is larger.
3. The operators $V(a_n)$ correspond to site-depending dilations

$$
\tilde{u}_n = e^{\epsilon a_n} u_n, \quad \tilde{v}_n = e^{\epsilon a_n} v_n.
$$

(6.2)

Invariance under two such one dimensional symmetry groups, generated by $\{V(a_{1,n}), V(a_{2,n})\}$, where $a_{1,n}$ and $a_{2,n}$ are two linearly independent functions of $n$, introduces the symmetry variable

$$
\omega_D \equiv (\xi_{n-1})^A (\xi_{n+1})^B (\xi_n)^C
$$

(6.3) as in eq.(5.11). Here all 6 variables are coupled together.

4. The pair of operators $Y_u(\lambda_n)$, $Y_v(\lambda_n)$ induces site-dependent (and time-dependent) shifts of the dependent variables.

$$
\tilde{u}_n = u_n + \epsilon \lambda_n(t), \quad \tilde{v}_n = v_n + \epsilon \lambda_n(t).
$$

(6.4)

The corresponding invariant variable again involves all 6 variables (see eq.(5.21)).

$$
\omega_T \equiv \lambda_{n-1} \xi_{n+1} - \lambda_n \xi_n - \lambda_{n+1} \xi_{n-1}.
$$

(6.5)

A special case of the variable $\omega_T$ is obtained setting $\lambda_n = \lambda_{n-1} = \lambda_{n+1} = 1$. This is the case of eq.(5.43), where

$$
\omega = \omega_S = \xi_{n+1} - \xi_n - \xi_{n-1}
$$

(6.6)

is invariant with respect to two such translations

$$
\tilde{u}_n = u_n + \epsilon_1 + \epsilon_2 t, \quad \tilde{v}_n = v_n + \epsilon_1 + \epsilon_2 t.
$$

(6.7)
\(\epsilon_1\) and \(\epsilon_2\) are group parameters and hence constants.

A continuation of this study is in progress. It involves several aspects.

The first is a study of the integrability properties of the equations that are completely specified by their symmetries. These are, first of all, those with infinite dimensional symmetry groups, namely

\[
\ddot{u}_n = u_{n+1} \frac{\xi_{n-1}}{\xi_n} h_n + u_n k_n, \\
\ddot{v}_n = v_{n+1} \frac{\xi_{n-1}}{\xi_n} h_n + v_n k_n,
\]

(6.8)

with \(h_n\) and \(k_n\) functions of \(t\) or constants. (see \(A_1(\infty)\) and \(A_2(\infty)\) in Table 1.)

Completely specified equations with finite dimensional symmetry algebras \(L\) are the following ones:

(i) \[
\ddot{u}_n = \left( u_{n+1} \frac{\xi_{n-1}}{\xi_n} p_n + u_n q_n \right) \omega_D^{-\frac{2}{\Delta}}, \\
\ddot{v}_n = \left( v_{n+1} \frac{\xi_{n-1}}{\xi_n} p_n + v_n q_n \right) \omega_D^{-\frac{2}{\Delta}},
\]

(6.9)

with \(\omega_D\) as in eq.(6.3), \(\Delta\) as in eq.(5.34). This is case \(A_3(7)\) of Table 1.

(ii) \[
\ddot{u}_n = \left( [u_{n+1} - u_{n-1}] p_n + (u_n - u_{n+1}) q_n \right) \omega_S^{-\frac{2}{2n+1}}, \\
\ddot{v}_n = \left( [v_{n+1} - v_{n-1}] p_n + (v_n - v_{n+1}) q_n \right) \omega_S^{-\frac{2}{2n+1}},
\]

(6.10)
with $\omega_S$ as in eq.(6.6), $p_n$, $q_n$, $a \neq -\frac{1}{2}$ constant. This is case $C_6(9)$ of Table 3.

(iii) For $a = 0$ eq.(6.10) is invariant under a 10 dimensional symmetry algebra, namely $C_{10}(10)$ of Table 3.

(iv)

\[
\begin{align*}
\ddot{u}_n &= (\xi_{n+1})^{-2A} (\xi_{n+1})^{-2B} (\xi_n)^{2A+2B-D} \left[ u_{n+1} \frac{\xi_n}{\xi_n} p_n + u_n q_n \right], \\
\ddot{v}_n &= (\xi_{n+1})^{-2A} (\xi_{n+1})^{-2B} (\xi_n)^{2A+2B-D} \left[ v_{n+1} \frac{\xi_n}{\xi_n} p_n + v_n q_n \right],
\end{align*}
\]

with $p_n$ and $q_n$ depending only on $n$. The constants $A$ and $B$ are given in eq.(5.12), $D$ in eq.(5.47).

A further task is to complete the classification, that is to treat the cases of other $sl(2,\mathbb{R})$ algebras and also of solvable symmetry algebras.

**ACKNOWLEDGEMENTS**

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References


Table 1: Series $A$ of symmetry algebras. The interaction has the form (5.3).

<table>
<thead>
<tr>
<th>No.</th>
<th>$L_C$</th>
<th>Restrictions on $h_n$ and $k_n$</th>
<th>Variables and comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1(3)$</td>
<td>-</td>
<td>-</td>
<td>$t, \xi_{n+1}, \xi_{n-1}, \xi_n$ (5.2)</td>
</tr>
<tr>
<td>$A_1(4)$</td>
<td>$V(a_n)$</td>
<td>-</td>
<td>${ t, \eta_{n+1}, \eta_{n-1} }$ (5.6) $t, \xi_n, \eta$ (5.7)</td>
</tr>
<tr>
<td>$A_1(5)$</td>
<td>$V(a_{1,n}), V(a_{2,n})$</td>
<td>-</td>
<td>$t, \eta$ (5.11)</td>
</tr>
<tr>
<td>$A_1(\infty)$</td>
<td>$V(a_{i,n}), i \in \mathbb{Z}^+$</td>
<td>-</td>
<td>$t$</td>
</tr>
<tr>
<td>$A_2(4)$</td>
<td>$T(b_n)$</td>
<td>-</td>
<td>$\zeta_{n+1}, \zeta_{n-1}, \zeta_n$ (5.9)</td>
</tr>
<tr>
<td>$A_2(5)$</td>
<td>$T(b_n), V(a_n)$</td>
<td>-</td>
<td>${ \rho_{n-1}, \rho_{n+1} }$ (5.13) $\rho_n, \sigma_n$ (5.14)</td>
</tr>
<tr>
<td>$A_2(6)$</td>
<td>$T(b_n), V(a_{1,n}), V(a_{2,n})$</td>
<td>-</td>
<td>$\eta$ (5.23)</td>
</tr>
<tr>
<td>$A_2(\infty)$</td>
<td>$T(b_n), V(a_{k,n}), k \in \mathbb{Z}^+$</td>
<td>$h_n, k_n$ constants</td>
<td>none</td>
</tr>
<tr>
<td>$A_3(5)$</td>
<td>$T(0), D(b_n)$</td>
<td>(5.16) or (5.18)</td>
<td>(5.17) or (5.19)</td>
</tr>
<tr>
<td>$A_3(6)$</td>
<td>$T(0), D(c_n), V(a_n)$</td>
<td>(5.25)</td>
<td>$\omega$ (5.25)</td>
</tr>
<tr>
<td>$A_3(7)$</td>
<td>$T(0), D(c_n), V(a_{1,n}), V(a_{2,n})$</td>
<td>(5.35)</td>
<td>none</td>
</tr>
</tbody>
</table>
Table 2: Series B of symmetry algebras. The algebra includes one pair $Y_u(\lambda_n),\ Y_v(\lambda_n)$. The interaction has the form (5.4).

<table>
<thead>
<tr>
<th>No.</th>
<th>Restrictions on $\lambda_n$, additional restrictions on $\phi_n$ and $\psi_n$</th>
<th>Variables and comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1(5)$</td>
<td>-</td>
<td>$t,\ \omega$ as in (5.21)</td>
</tr>
<tr>
<td>$B_4(6)$</td>
<td>$\lambda_n = e^{a_n t}, T(a_n)$</td>
<td>(5.30) $\omega$ (5.30)</td>
</tr>
<tr>
<td>$B_3(6)$</td>
<td>$\lambda_n = e^{(a_n-1)t}, T(a_n)$</td>
<td>(5.33) $\omega$ (5.30)</td>
</tr>
</tbody>
</table>
Table 3: Series C symmetry algebras. The algebras contain $sl(2, \mathbb{R})_1 \ Y_u(1), \ Y_v(1), \ Y_u(t), \ Y_v(t)$ and possibly additional elements. The interaction is as in eq.(5.42).

<table>
<thead>
<tr>
<th>No</th>
<th>Additional elements</th>
<th>Conditions on $\phi_n$ and $\psi_n$</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1(7)$</td>
<td>-</td>
<td>-</td>
<td>$\omega, \ t$ (5.43)</td>
</tr>
<tr>
<td>$C_2(8)$</td>
<td>$T(a)$</td>
<td>-</td>
<td>$\eta = \omega e^{-2at}$</td>
</tr>
<tr>
<td>$C_3(8)$</td>
<td>$D(a)$</td>
<td>$\phi_n = t^{-2}r_n(\eta), \ \psi_n = t^{-2}s_n(\eta)$</td>
<td>$\eta = \omega t^{-(2a+1)}$</td>
</tr>
<tr>
<td>$C_4(8)$</td>
<td>$R(b)$</td>
<td>$\phi_n = (t^2 + 1)^{-2}r_n(\eta), \ \psi_n = (t^2 + 1)^{-2}s_n(\eta)$</td>
<td>$\eta = \omega(t^2 + 1)^{-1} e^{-2b \arctan t}$</td>
</tr>
<tr>
<td>$C_6(9)$</td>
<td>$T(0), D(a)$</td>
<td>$\phi_n = k_n \omega^{-\frac{2}{2a+1}}, \ \psi_n = p_n \omega^{-\frac{2}{2a+1}}$</td>
<td>none</td>
</tr>
<tr>
<td>$C_6(9)$</td>
<td>$T(0)$, $D(0)$, $P(0)$</td>
<td>$\phi_n = k_n \omega^{-2}, \ \psi_n = p_n \omega^{-2}$</td>
<td>none</td>
</tr>
</tbody>
</table>

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Table 4: Series D of symmetry algebras. The algebra contains $sl(2,\mathbb{R})_1 \oplus sl(2,\mathbb{R})_2$. The interaction has the form (5.28).

<table>
<thead>
<tr>
<th>No</th>
<th>Additional elements in $L_C$</th>
<th>Conditions on $p_n$ and $q_n$</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1(6)$</td>
<td>-</td>
<td>-</td>
<td>$\chi_{n+1}, \chi_{n-1}$ as in (5.28)</td>
</tr>
<tr>
<td>$D_2(7)$</td>
<td>$V(a_n)$</td>
<td>(5.37)</td>
<td>$\eta$ as in (5.37)</td>
</tr>
<tr>
<td>$D_3(8)$</td>
<td>$V(a_{1,n}), V(a_{2,n})$</td>
<td>(5.46)</td>
<td>-</td>
</tr>
</tbody>
</table>