Supersymmetry and algebraic deformations

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Abstract. We describe a class of algebraically solvable SUSY models by considering the deformation of invariant polynomial flags by means of the Darboux transformation. The algebraic deformations corresponding to the addition of a bound state to a shape-invariant potential are particularly interesting. The polynomial flags in question are indexed by a deformation parameter \( m = 1, 2, \ldots \) and lead to new algebraically solvable models. We illustrate these ideas by considering deformations of the hyperbolic Pöschl-Teller potential.

PACS numbers: 03.65.Fd, 03.65.Ge

1. Introduction

Our purpose in this paper is to show how new classes of exactly solvable supersymmetric quantum mechanical Hamiltonians arise in a natural fashion from the application of the Darboux transformation to classes of second order linear differential operators that preserve flags of vector spaces generated by univariate polynomials. These new Hamiltonians and their bound states have closed analytic expressions in terms of elementary functions, and their qualitative behavior is both natural and significant from a physical point of view.

Recall that the Darboux transformation starts from the knowledge of a formal eigenfunction of a Schrödinger operator, which is used to factorize it as a product of first-order operators. Depending on whether this formal eigenfunction or its reciprocal are square integrable, one obtains forward or backward Darboux transformations, in which the supersymmetric partner Hamiltonian is obtained by reversing the order of these factors. The principle of our approach is to consider only those factorizations for which the effect of the Darboux transformation on functions is to map polynomials to polynomials. These are given a simple characterization in our paper in terms of the starting formal eigenfunction. We will refer to this transformation as the algebraic Darboux transformation. The new Hamiltonians obtained in this fashion will be exactly solvable in the precise algebraic sense that they will also admit complete invariant flags of polynomial subspaces.

When a parametrized family of potentials is closed with respect to the forward Darboux transformation, it is said to be shape-invariant [1] and in this case the
Supersymmetry and algebraic deformations

The iteration of the transformation furnishes a complete description of the spectrum and eigenfunctions. For the shape invariant potentials, the underlying invariant flag of polynomials is the full polynomial module. Thus, to obtain deformations of a shape-invariant potential one must consider the two-parameter family of backwards Darboux transformations. These were first applied to the harmonic oscillator in [2], while the general theory was developed in [3,4]. However, as noted in [6], the general form of the deformed potential can only be expressed by a formal power series, or as the integral of eigenfunctions of the original Hamiltonian – in contrast to the original potential, which is an elementary function, with bound states also described by elementary functions. Our main emphasis in this paper is to obtain examples of exactly solvable potentials which lie outside the shape-invariant class, which can be expressed in closed analytic form, and which have qualitative properties that make them relevant to the description of physically realistic situations. These are obtained by the application of the backward Darboux transformation to shape invariant potentials corresponding to special values of the parameters. There is a countable infinity of algebraic backwards transformations of a shape-invariant potential [7], indexed by an integer \(m\). We will show the precise manner in which the \(m\text{th}\) algebraic backward transformation deforms the invariant polynomial flag of a shape invariant potential, and we calculate the explicit basis of the deformed flags for the cases \(m = 1\) and \(m = 2\). As an illustrative example we discuss in detail the algebraic deformation of the hyperbolic Pöschl-Teller potential.

2. Darboux transformations

2.1. The self-adjoint case.

Consider the Schrödinger operator

\[
H = -\partial_{xx} + u,
\]

where \(u(x), x \in \mathbb{R}\) is continuous, real-valued and bounded from below. Consequently, the restriction of \(H\) to a certain dense subspace \(D(H) \subset L^2(\mathbb{R})\) is a self-adjoint operator. Consider a formal eigenfunction

\[
H[\phi] = \lambda_0 \phi.
\]

The key idea of the supersymmetric or Darboux transformation is the fact that to every \(\phi\) there corresponds a factorization of \(H\) as

\[
H - \lambda_0 = A^* A,
\]

where

\[
A = \partial_x - (\log \phi)_x, \quad A^* = -\partial_x - (\log \phi)_x.
\]

We shall refer to \(\phi\) as the factorization function, and to \(\lambda_0\) as the factorization energy. The supersymmetric partner potential is the operator defined by the commutation of the factors

\[
\hat{H} = -\partial_{xx} + \hat{u} = AA^* + \lambda_0, \quad \hat{u} = u - 2(\log \phi)_{xx}.
\]

The transformed potential \(\hat{u}\) is continuous if and only if \(\phi\) is non-vanishing, which we assume from here on. In this way, \(\hat{H}\) is self-adjoint and semi-bounded on some dense domain \(D(\hat{H})\). The operators \(H\) and \(\hat{H}\) satisfy the intertwining relation

\[
AH = \hat{H} A,
\]
which implies the following relation between the eigenfunctions of the two operators:

\[ H[\psi] = \lambda \psi, \quad \hat{H}[\hat{\psi}] = \lambda \hat{\psi}, \quad \hat{\psi} = A[\psi]. \tag{6} \]

The spectral properties of this transformation are governed by one of the following three possibilities [4, 5].

(i) **Forward transformation:** \( \phi \) is square integrable (and since it is nodeless, it must be the ground-state wavefunction of \( H \)). The operator \( A \) maps \( D(H) \) onto \( D(\hat{H}) \), with a 1-dimensional kernel. The \( n^{th} \) bound state of \( H \) is mapped to the \((n - 1)^{st} \) bound state of \( \hat{H} \). Correspondingly, the transformed spectrum differs from the spectrum of \( H \) by the removal of \( \lambda_0 \), the lowest eigenvalue.

(ii) **Backward transformation:** \( \phi^{-1} \) is square integrable. The operator \( A \) maps the \( n^{th} \) bound state of \( H \) to the \((n + 1)^{st} \) bound state of \( \hat{H} \). It is one-to-one on \( D(H) \), but not onto \( D(\hat{H}) \); the new ground state is not in the image. The spectrum of \( \hat{H} \) differs from that of \( H \) by the addition of a lowest eigenvalue, namely \( \lambda \), with the ground state given by \( \phi^{-1} \). A 2-parameter family (energy and shape parameter) of backward transformations exist for every \( \lambda \) strictly smaller than the infimum of the spectrum of \( H \).

(iii) **Isospectral transformation:** neither \( \phi \) nor \( \phi^{-1} \) are square integrable. The operator \( A \) defines a linear isomorphism from \( D(H) \) to \( D(\hat{H}) \). It transforms the \( n^{th} \) bound state of \( H \) to the \( n^{th} \) bound state of \( \hat{H} \). Two isospectral Darboux transformations exist for every \( \lambda \) strictly smaller than the infimum of the spectrum of \( H \).

2.2. The general form and covariance.

In this paragraph we will consider Darboux transformations of an arbitrary second order operator, the general form of which is

\[ T = p \partial_{zz} + q \partial_z + r, \tag{7} \]

where we assume that \( p(z) < 0 \) on the domain of interest. The above operator is related to a Schrödinger operator by the change of variables

\[ x = \int (-p)^{-\frac{1}{2}} \, dz, \tag{8} \]

and gauge transformation

\[ H = e^{\rho} T e^{-\rho}, \quad \rho = \int \frac{1}{2} p^{-1} (q - \frac{1}{2} p z) \, dz. \tag{9} \]

The relation to the potential is given by

\[ u = \frac{1}{4} p_{zz} - \frac{1}{2} q_z - \frac{1}{4} p^{-1} (q - \frac{1}{2} p z)(q - \frac{3}{2} p z) + r \tag{10} \]

Since gauge transformations and changes of variable are homomorphisms of the ring of differential operators, the Darboux transformation is covariant with respect to these operations.

Although it is customary to work in the Schrödinger gauge as in Section 2.1, the Darboux transformation can be defined relative to a general coordinate and choice of gauge, as shown below. Indeed, let

\[ T[\phi] = \lambda_0 \phi, \tag{11} \]
Supersymmetry and algebraic deformations

be a factorization eigenfunction. Writing

\[
(\log \phi)_z = \frac{a}{b},
\]

we have the following factorization of \( T \):

\[
T = BA + \lambda_0,
\]

where

\[
A = b\partial_z - a = b(\partial_z - (\log \phi)_z),
\]

\[
B = \frac{p}{b} \left( \partial_z + \frac{a-bz}{b} + \frac{q}{p} \right) = (p\partial_z + p(\log \phi)_z + q)b^{-1}.
\]

We define the partner operator to be

\[
\hat{T} = AB + \lambda_0,
\]

and observe that the following intertwining relation holds

\[
\hat{T}A = AT.
\]

A particular case occurs when \( q = 1/2 p_z \), i.e. when the operator is in the self-adjoint gauge. Taking

\[
b = (-p)^{1/2}, \quad a = (-p)^{1/2}(\log \phi)_z,
\]

we have \( B = A^* \), and consequently \( \hat{T} \) is equivalent, after a change of variables, to the self-adjoint factorization \( T \). We also note that the mapping \( T \mapsto \hat{T} \) is not canonical, but rather covariant with respect to gauge transformations, since two different choices of the denominator in \( \log \phi)_z = a/b = a'/b' \),

will lead to partner operators \( \hat{T} \) and \( \hat{T}' \) which are related by a gauge transformation:

\[
b^{-1} \hat{T} \hat{b} - \lambda_0 = (b')^{-1} \hat{T}' \hat{b}' - \lambda_0 = (\partial_z - (\log \phi)_z)(p\partial_z + q + p(\log \phi)_z).
\]

To effect an inverse transformation, we factorize \( \hat{T} \) with

\[
\hat{\phi} = b \exp \left( - \int \left( \frac{a}{p} + \frac{a}{b} \right) dz \right)
\]

as the factorization function. A simple calculation shows that

\[
\hat{T}[\hat{\phi}] = \lambda_0 \hat{\phi},
\]

and \( \hat{T} \) has the following form

\[
\hat{T} = p\partial_{zz} + \hat{q}\partial_z + \hat{r},
\]

where

\[
\hat{q} = q + p_z - 2p(\log b)_z,
\]

\[
\hat{r} = -p(\log b)_{zz} + p(\log b)_z^2 - (p_z + q)(\log b)_z
\]

\[
- \frac{2pa^2}{b^2} + (p_z - 2q)\frac{a}{b} + q_z + 2\lambda_0 - r.
\]

Thus, taking

\[
\hat{a} = p(b_z - a) - qb, \quad \hat{b} = pb,
\]

we have \( (\log \hat{\phi})_z = \hat{a}/\hat{b} \), and from \( \text{13} \), \( \text{15} \) we obtain

\[
\hat{T} = \hat{B}\hat{A} + \lambda_0,
\]
Supersymmetry and algebraic deformations

where

$$\hat{A} = \hat{b}\partial_z - \hat{a} = b p \left( \partial_z + \frac{a - b_z}{b} + \frac{q}{p} \right) = b^2 B,$$

(22)

$$\hat{B} = \frac{p}{b} \left( \partial_z + \frac{\hat{a} - \hat{b}_z}{\hat{b}} + \frac{\hat{q}}{p} \right)$$

(23)

$$= \frac{1}{b} \left( \partial_z - \frac{a}{b} - \frac{2b_z}{b} \right) = b \left( \partial_z - \frac{a}{b} \right) b^{-2} = Ab^{-2}.$$  

It follows that

$$\hat{T} = \hat{A}\hat{B} + \lambda_0 = b^2 T b^{-2}.$$  

2.3. Algebraic factorizations.

We will say that a second-order differential operator is exactly solvable by polynomials (P.E.S.) if it is equivalent, by a change of variable and a gauge transformation, to a second-order operator $T$ that preserves an infinite flag of finite-dimensional polynomial subspaces

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \subset \ldots \subset \mathcal{M} = \bigcup_n \mathcal{M}_n.$$  

(24)

As part of this definition we include the following assumptions:

(E1) There is a fixed polynomial codimension, which we will call $m$. To be more precise, we assume that each $\mathcal{M}_n$ is an $n$-dimensional subspace of

$$P_{n+m-1} = \langle 1, z, z^2, \ldots, z^{n+m-1} \rangle.$$  

(E2) There is no spectral degeneracy. The action of $T$ is upper-triangular relative to a basis adapted to the above flag, and hence possesses an infinite list of eigenpolynomials. We assume that the corresponding eigenvalues are distinct.

This definition is similar to the definition of exact solvability introduced in [8]. Of particular interest is the subclass of P.E.S. operators for which $\mathcal{M}_1 = \mathbb{R}$. If this condition holds, we will say that the operator satisfies the algebraic ground state condition.

We will now consider the following question: suppose that $T$ is a P.E.S. operator with invariant flag (24), what are the conditions on a factorization function $\phi$ such that the partner operator $\hat{T}$ is also P.E.S.? To answer this question, we define a factorization eigenfunction (11) to be of algebraic type whenever

$$\frac{\phi_z}{\phi} = \frac{a}{b}$$

is a rational function, where without loss of generality the polynomials $a = a(z)$ and $b = b(z)$ are assumed to be relatively prime. This definition is motivated by the following observation.

**Proposition 1** The factorization function $\phi$ is of algebraic type, if and only if the operator

$$A = b\partial_z - a = b(\partial_z - (\log \phi)_z),$$

transforms polynomials into polynomials.
We would like to focus on those Darboux transformations that preserve the P.E.S. character, from here on called algebraic Darboux transformations. According to Proposition 1 these are precisely the ones in which the factorization function $\phi$ is of algebraic type. Let us describe the invariant polynomial flag of the partner operator in each of the three cases discussed in Section 2.1.

(A1) **Algebraic forward transformation:** In this case $\mathcal{M}_1 = \langle \phi \rangle$, where $\phi$ is the factorization function. The operator

$$A = \phi \partial_z - \phi_z$$

dele tes the ground state, and therefore the invariant flag of the partner is

$$\hat{\mathcal{M}}_n = A[\mathcal{M}_{n+1}].$$

(A2) **Algebraic backward transformation:** In this case, the new ground state $\hat{\phi}$, given by (19), is a rational function. Writing $\hat{\phi} = \tilde{a}/\tilde{b}$, we see that the first condition is true if and only if there exist polynomials $\tilde{a} = \tilde{a}(z)$, and $\tilde{b} = \tilde{b}(z)$ such that

$$\frac{b_z - q}{p} - \frac{a}{b} = \frac{\tilde{a}}{\tilde{b}} - \frac{\tilde{b}_z}{\tilde{b}}, \quad (25)$$

The partner flag is given by

$$\hat{\mathcal{M}}_1 = \langle \hat{\phi} \rangle, \quad \hat{\mathcal{M}}_{n+1} = \hat{b} A[\mathcal{M}_n] \oplus \langle \hat{\phi} \rangle.$$

(A3) **Isospectral transformation:** In this case both $A$ and $B$ are linear isomorphisms, and we have

$$\hat{\mathcal{M}}_n = A[\mathcal{M}_n].$$

Of particular interest is the case where the partner operators represent an algebraic forward/backward transformation pair, and where both operators satisfy the algebraic ground state condition.

**Proposition 2** Let $T, \hat{T}, \phi, a, b$ be as above, and suppose that $T$ satisfies the algebraic ground state condition. The following are equivalent:

(i) $T \mapsto \hat{T}$ is an algebraic backward transformation, with $\hat{T}$ satisfying the algebraic ground state condition.

(ii) $paz + (r - \lambda_0)b = 0$.

(iii) $\hat{\phi}$ is a constant.

**Proof.** Without loss of generality, $T$ annihilates the constants, and it is therefore of the form

$$T = p \partial_{zz} + q \partial_z.$$ 

The implication (iii) $\Rightarrow$ (i) follows directly from the definitions. In order to prove the converse, suppose that (i) holds. By (15), (19) we have then

$$B = \frac{p}{b} \left( \partial_z - \frac{\hat{\phi}}{\phi} \right) = \left( \frac{p\hat{\phi}}{b} \partial_z \right) \hat{\phi}^{-1}. \quad (26)$$

Hence,

$$T = BA + \lambda_0 = \left( \frac{p\hat{\phi}}{b} \partial_z \right) \left( \frac{b}{\phi} \partial_z - \frac{a}{\phi} \right) + \lambda_0$$

$$= p \partial_{zz} + q \partial_z - \frac{p\hat{\phi}}{b} \left( \frac{a}{\phi} \right)_z + \lambda_0. \quad (27)$$
Note that $A[1] = -a$, and therefore the first two invariant subspaces of the partner operator $AB$ are given by
\[
\langle \hat{\phi} \rangle = \hat{\mathcal{M}}_1 \quad \text{and} \quad \langle \hat{\phi}, a \rangle = \hat{\mathcal{M}}_2.
\]
Note that $\hat{T}$ satisfies the algebraic ground state condition if and only if $a/\hat{\phi}$ is a polynomial. Also, by (26) (27) we have
\[
\left( \frac{a}{\phi} \right)_z = \frac{b}{p\phi}.
\]
Since (A2) implies that $\lambda_0 \neq 0$, $\hat{\phi}$ divides both $a$ and $b$, and therefore $\hat{\phi}$ must be a constant.

Finally, let us show that (iii) is equivalent to (ii). By (19), (iii) is true if and only if
\[
\frac{b_z - a}{b} - \frac{q}{p} = 0.
\]
By assumption,
\[
T[\phi] = p\phi_{zz} + q\phi_z = \lambda_0 \phi, \quad \text{and} \quad \phi_z = \frac{a}{b} \phi,
\]
and therefore
\[
\frac{a_z}{b} - \frac{a b_z}{b} + \frac{q}{p} = \lambda_0,
\]
or equivalently,
\[
p a_z - \lambda_0 b = p a \left( \frac{b_z - a}{b} - \frac{q}{p} \right),
\]
so that (ii) holds if and only if (28) does.

3. Algebraic deformations of shape invariant potentials.

3.1. Shape invariance.

Let us recall that a parameterized potential is called shape-invariant [1] if the forward Darboux transformation preserves the form of the potential while altering the value of the parameters. In the preceding section, we pointed out that the Darboux transformation is covariant with respect to arbitrary changes of gauge and variable. As a consequence, the notion of shape-invariance makes perfect sense for general second-order operators, and not just for operators in Schrödinger form. Thus, we will adapt the usual definition and say that a parameterized family of P.E.S. operators is shape-invariant if that family is closed with respect to the forward (A1) Darboux transformation.

We now describe an important class of shape-invariant, P.E.S. operators. We define the standard polynomial flag to be:
\[
\mathbb{R} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n \subset \ldots, \quad \mathcal{P}_n = \langle 1, z, \ldots, z^n \rangle. \tag{29}
\]
The general form of a second-order operator $T$ that preserves the standard flag is
\[
T = p\phi_{zz} + q\phi_z + r, \tag{30}
\]
where $p = p(z)$ and $q = q(z)$ are, respectively, second and first degree polynomials, and where $r$ is a constant. The family of operators described by (30) is shape invariant in
the above sense. The ground state is given by \( \phi = 1 \) with \( \lambda_0 = r \), and the factorization is simply

\[
T = (p\partial_z + q)\partial_z + \lambda_0.
\]

The partner operator

\[
\hat{T} = \partial_z(p\partial_z + q) + \lambda_0 = p\partial_{zz} + \hat{q}\partial_z + \hat{r},
\]

retains the form (30), with

\[
\hat{q} = p_2 + q, \quad \hat{r} = q_2 + r.
\]

The corresponding non-singular potential forms — see (8) (9) (10) for the transformation formulas and [7] for their derivation — are shown in Table 1. These are the well-known, shape-invariant potential families: the harmonic oscillator (I), the Morse potential (II), and the hyperbolic Pöschl-Teller potentials (III). Since potentials (I) and (III) are even functions, the corresponding eigenfunctions have a well-defined parity. Consequently, these potentials possess two algebraic sectors, i.e. they are exactly solvable by polynomials in two distinct ways (see [10] for an algebraic explanation of potentials with multiple algebraic sectors). The even sector corresponds to an even gauge factor, and the odd sector to an odd gauge factor. The parity of the algebraic sectors is reversed by a Darboux transformation.

<table>
<thead>
<tr>
<th></th>
<th>I_e</th>
<th>I_o</th>
<th>II</th>
<th>III_e</th>
<th>III_o</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>(-4z)</td>
<td>(-4z)</td>
<td>(-z^2)</td>
<td>(z(1 - z))</td>
<td>(z(1 - z))</td>
</tr>
<tr>
<td>( q )</td>
<td>(4z - 2)</td>
<td>(4z - 6)</td>
<td>((2A - 1)z - 1)</td>
<td>((A - \frac{1}{2})z + 1 - A)</td>
<td>((A - \frac{1}{2})z + 1 - A)</td>
</tr>
<tr>
<td>( r )</td>
<td>1</td>
<td>3</td>
<td>(-A^2)</td>
<td>(-\left(\frac{A}{2} - \frac{1}{4}\right)^2)</td>
<td>(-\left(\frac{A}{2} - \frac{3}{4}\right)^2)</td>
</tr>
<tr>
<td>( e^\rho )</td>
<td>(e^{-\frac{x^2}{2}})</td>
<td>(xe^{-\frac{x^2}{2}})</td>
<td>(e^{-\frac{x}{2} - A^2} \cosh^2(\frac{x}{2}))</td>
<td>(\sinh^2(\frac{x}{2}) \cosh^2(\frac{x}{2}))</td>
<td></td>
</tr>
<tr>
<td>( z(x) )</td>
<td>(x^2)</td>
<td>(x^2)</td>
<td>(1 - (A + \frac{1}{2})e^{-x})</td>
<td>(\frac{1}{4}(1 - A^2))</td>
<td>(\frac{1}{4}(1 - A^2))</td>
</tr>
<tr>
<td>( U )</td>
<td>(x^2)</td>
<td>(x^2)</td>
<td>(\frac{1}{4}e^{-2x} \cosh^2(\frac{x}{2}))</td>
<td>(\frac{1}{4}(1 - A^2))</td>
<td>(\frac{1}{4}(1 - A^2))</td>
</tr>
</tbody>
</table>

Table 1. Shape-invariant potentials on the line

3.2. Deformations of the standard flag.

Let \( a = a(z), b = b(z) \) be relatively prime polynomials, and let \( g = g(z) \) be a polynomial that divides \( a, b \) and \( b z - a \). Consider the differential operators

\[
B = g^{-1}\partial_z, \quad A = b\partial_z - a, \quad (31)
\]

and note that, by assumption, the second order operator

\[
T = BA = p\partial_{zz} + q\partial_z - a\partial_z g^{-1}, \quad (32)
\]

where

\[
p = bg^{-1}, \quad q = (bz - a)g^{-1},
\]

has polynomial coefficients. We will say that \( A, B \) constitute a deformation pair of order \( m = \text{deg}(g) \) if \( T \) leaves invariant \( M_n = P_{n-1} \) for all \( n \), i.e. if \( T \) leaves invariant the standard polynomial flag. Deformation pairs are of interest because they provide non-trivial examples of exactly solvable Hamiltonians outside the shape-invariant class. As usual, we define a partner operator

\[
\hat{T} = AB = p\partial_{zz} + \hat{q}\partial_z, \quad \hat{q} = -(bg + a)g^{-1}, \quad (33)
\]
and a partner flag
\[ \mathbb{R} = \hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}}_2 \subset \hat{\mathcal{M}}_3 \subset \ldots, \quad \hat{\mathcal{M}}_n = A[\mathcal{P}_{n-2}] \oplus \mathbb{R}, \]
which we will refer to as a deformation of the standard polynomial flag.

**Proposition 3** Every \( \hat{\mathcal{M}}_n \) is a codimension \( m \) subspace of \( \mathcal{P}_{n+m-1} \).

**Proof.** By assumption, \( a_z g^{-1} \) is constant, and \( \text{deg}(bg^{-1}) \leq 2 \). This implies that \( \text{deg}(a) = m + 1 \) and that \( \text{deg}(b) \leq m + 2 \). Therefore, \( \mathcal{M}_n \subset \mathcal{P}_{n+m-1} \). Since \( a, b \) are relatively prime, \( A \) does not annihilate any polynomial, and hence \( \dim \hat{\mathcal{M}}_n = n \). \( \square \)

**Proposition 4** The partner operator \( \hat{T} \) is P.E.S.

**Proof.** The operator \( \hat{T} \) preserves the deformed flag because of the intertwining relation
\[ \hat{T} A = AT, \]
and because \( \hat{T} \) annihilates \( \mathbb{R} = \hat{\mathcal{M}}_1 \) by construction. Condition (E1) is true by the preceding proposition. We noted above that \( A \) and hence that \( T \) does not annihilate any polynomial. Hence 0 is not an eigenvalue of \( T \), which proves condition (E2).

Let us also note that a deformation pair satisfies the conditions of Proposition 2, and in particular \( T \mapsto \hat{T} \) is corresponds to an algebraic backward Darboux transformation.

**Proposition 5** The deformed subspaces \( \hat{\mathcal{M}}_n \) can be characterized as the subspace of \( \mathcal{P}_{n+m-1} \) consisting of all polynomials \( f = f(z) \) such that \( g \) divides \( f_z \).

**Proof.** First, note that for
\[ f = A[h], \quad h \in \mathcal{M}_{n-1} = \mathcal{P}_{n-2}, \quad f \in \hat{\mathcal{M}}_n \]
we have that \( g^{-1} f_z = T[h] \), which proves that \( f_z \) is divisible by \( g \). In order to prove the converse, let us note that the subspace of all \( f \in \mathcal{P}_{n+m-1} \) such that \( f_z \) is divisible by \( g \) is \( n \)-dimensional. However, \( \dim \mathcal{M}_n = n \) by Proposition 3 which proves the claim. \( \square \)

We will now show an explicit basis of \( \hat{\mathcal{M}}_n = n \) in the cases \( m = 1 \) and \( m = 2 \). In the first instance, since the subspaces \( \mathcal{P}_n \) are invariant with respect to translations, we may without loss of generality assume that \( g(z) = z \). By (34), necessarily, \( b = pz \), where \( p = p_2 z^2 + p_1 z + p_0 \) is a polynomial of degree 2 or less. In order for \( g \) to divide \( a_z + b_z - a \) we must have
\[ A = (p_2 z^2 + p_1 z + p_0) z \partial_z - (a_2 z^2 + p_0), \]
where \( a_2, p_0, p_1, p_2 \) are arbitrary real numbers. Let us also assume that \( a_2 \neq 0 \) and that \( a_2/p_2 \) is not a positive integer. If these generic conditions hold, then the subspaces of the partner flag are given by
\[ A[\mathcal{P}_{n-2}] \oplus \mathbb{R} = \hat{\mathcal{M}}_n = (1, z^2, z^3, \ldots, z^n) \]
The above monomial-generated subspace is exceptional in that it admits a seven dimensional vector space of second-order operators that preserve it [11], and consequently can be used to construct novel instances of exactly solvable and quasi-exactly solvable operators [7, 10].

Turning to the case \( m = 2 \), we limit our discussion to the generic case of \( g(z) \) with distinct roots. By scaling and translating \( z \), as necessary, we may assume, without
loss of generality, that \( g = z^2 - 1 \). By \( \Box \) in order for \( g \) to divide \( a_z \) and \( b_z - a \) we must have

\[
A = (p_2 z^2 + p_1 z + p_0) (z^2 - 1) \partial_z + (p_2 + p_0) (z^3 - 3z) - 2p_1,
\]

where \( p_0, p_1, p_2 \) are arbitrary real numbers. Let us also assume that \( p_2 + p_0 \neq 0 \) and that \(-p_0/p_2\) is not a positive integer. If these generic conditions hold, then the subspaces of the partner flag are given by

\[
A[P_n - z] \oplus \mathbb{R} = \mathcal{M}_n = (1, \pi_3(z), \pi_4(z), \ldots \pi_{n+1}(z)),
\]

where

\[
\pi_{2k+1}(z) = z^{2k+1} - (2k+1)z, \quad \pi_{2k}(z) = z^{2k} - k z^2.
\]

The above polynomials \( \pi = \pi(z) \) have the property that \( \pi_z \) is divisible by \( z^2 - 1 \). The resulting polynomial subspaces \( \mathcal{M}_n \) are preserved by the following second-order operators:

\[
\begin{align*}
T_3 &= z^3 \partial_{zz} + \left( (1-n) z^2 - 5 + n - \frac{4}{z^2 - 1} \right) \partial_z, \\
T_2 &= (z^2 - 1) \partial_{zz} - 2z \partial_z, \\
T_1 &= z \partial_{zz} - 2 \left( 1 + \frac{2}{z^2 - 1} \right) \partial_z, \\
T_0 &= \partial_{zz} + \left( z - \frac{4z}{z^2 - 1} \right) \partial_z.
\end{align*}
\]

3.3. Algebraic deformations of the hyperbolic Pöschl-Teller potential

The hyperbolic Pöschl-Teller potential [14], which includes the class of reflectionless 1-soliton potentials [15], has the form

\[
U_{PT}(x) = \frac{1}{4} \left( \frac{1}{2} - \alpha^2 \right) \text{sech}^2(\frac{x}{\alpha}).
\]

(37)

The general solution [16, Sec. 2.9] of the corresponding Schrödinger equation

\[
H_{PT}(\phi) = -\phi_{xx} + U_{PT} \phi = -k^2 \phi
\]

can be given as

\[
\phi_{PT}(x; k, C_0, C_1) = \cosh(\frac{x}{\alpha})^{\frac{1}{2} - \alpha} \left\{ C_0 \ 2F_1\left( -\frac{\alpha}{2} + \frac{1}{4}; \frac{1}{2} + k, -\frac{\alpha}{2} + \frac{1}{2} - k, \frac{1}{2}; -\sinh^2(\frac{x}{\alpha}) \right) \\
+ C_1 \sinh(\frac{x}{\alpha}) \ 2F_1\left( -\frac{\alpha}{2} + \frac{3}{4}; \frac{1}{2} + k, -\frac{\alpha}{2} + \frac{3}{4} - k, \frac{3}{4}; -\sinh^2(\frac{x}{\alpha}) \right) \right\},
\]

where \( 2F_1(a, b; c; z) \) also denotes the analytic continuation of the hypergeometric function to \( \text{Re}(z) < 0 \). For \( \alpha > 1/2 \), the potential \( \Box \) has \( \lfloor \alpha - \frac{1}{2} \rfloor \) bound states

\[
\psi_{\nu,i}(x), \quad 0 \leq i < \alpha - \frac{1}{2}.
\]

The even bound states are given by [7]

\[
\psi_{\nu,2j}(x) \propto \phi_{PT}(x; \frac{\nu}{2} - j - \frac{1}{4}, 1, 0) \propto \cosh(\frac{x}{\alpha})^{\frac{1}{2} - \alpha} \ P_{j}^{(-\frac{1}{2}, -\alpha)}(\cosh x).
\]

(38)

The odd ones are given by

\[
\psi_{\nu,2j+1}(x) \propto \phi_{PT}(x; \frac{\nu}{2} - j - \frac{3}{4}, 0, 1) \propto \sinh(\frac{x}{\alpha}) \cosh(\frac{x}{\alpha})^{\frac{1}{2} - \alpha} \ P_{j}^{(-\frac{1}{2}, -\alpha)}(\cosh x),
\]

(39)
where \( P^{(a,b)}_j(z) \) are the Jacobi polynomials. We focus on deformations of potentials with bound states only, i.e., we must take \( \alpha > \frac{1}{2} \). In order to have a well-defined backwards Darboux transform of the hyperbolic Pöschl-Teller potential, we must consider the solutions \( \phi_{PT} \) which correspond to an energy below the spectral minimum and are nowhere vanishing. Since the spectral minimum is \( -(\frac{1}{2} - \alpha)^2 \) we take \(|k| > \frac{1}{2} - \alpha\). Now, the two-parameter family of backward Darboux transformations is given by the transformation functions:

\[
\phi_{PT}(x; k, 1, t), \quad |t| \leq 2 \frac{\Gamma(\frac{3}{4} + k - \frac{\alpha}{2})\Gamma(\frac{3}{4} + k + \frac{\alpha}{2})}{\Gamma(\frac{1}{4} + k - \frac{\alpha}{2})\Gamma(\frac{1}{4} + k + \frac{\alpha}{2})},
\]

with the extreme values of the shape parameter \( t \) corresponding to an isospectral transformation [7]. In general, for an arbitrary value of the energy parameter \( k \) the partner potential will be defined formally by a power series. However, for specific values of \( k \) the log-derivative of \( \phi_{PT} \) will be a polynomial in \( \cosh x \) thus giving rise to an algebraic deformation. It can be shown that these algebraic deformations occur precisely for the following countable subset of (40): 

\[
\phi_{PT}^{(m)}(x) = \phi_{PT}(x, -\frac{\alpha}{2} - \frac{1}{4} - m, 1, 0)
\]

\[ \times \cosh(\frac{x}{2})^{2 + \alpha} P_m(-\frac{\alpha}{2}, \alpha)(\cosh(x)), \]

The resulting deformed potentials, as given by (41), have the form

\[
U_{PT}^{(m)}(x) = -\frac{1}{2}(\alpha + \frac{1}{2})(\alpha + \frac{3}{2}) \text{sech}^2(\frac{x}{2}) - 2 \left( \log P_m(-\frac{\alpha}{2}, \alpha)(\cosh x) \right)_{xx},
\]

and have been plotted in Figure 1.

More specifically, the first \((m = 1)\) and second \((m = 2)\) deformations have the following forms:

\[
U_{PT}^{(1)}(x) = U_{PT}^{(0)}(x) + \frac{2\alpha + 1}{z_1} - \frac{4(\alpha + 1)}{z_1^2},
\]

\[
U_{PT}^{(2)}(x) = U_{PT}^{(0)}(x) + \frac{2\alpha + 1}{z_1} - \frac{4(\alpha + 1)}{z_1^2},
\]
where
\[ z_1 = \frac{1}{2}((2\alpha + 3) \cosh x - (2\alpha + 1)), \]
and
\[ U^{(2)}_i(x) = U^{(0)}_i(x) + \frac{(2\alpha + 1) \left( \beta (z_2^3 + 3z_2) - 2z_2^2 - 2 \right)}{(z_2^3 - 1)^2}, \]
where
\[ z_2 = \frac{1}{4} \beta \left( (2\alpha + 7) \cosh x - (2\alpha + 1) \right), \quad \beta = \sqrt{\frac{2\alpha + 5}{3\alpha + 6}}. \]
The algebraic backwards Darboux transformation corresponds to the first order operator
\[ A^{(m)}_{PT} = \partial_x - \left( \log \phi^{(m)}_{PT} \right)_x \]
\[ = \partial_x - \frac{1}{2} \left( m + \alpha + \frac{1}{2} \right) \sinh x \frac{P_{m-1}^{(\frac{1}{2},\alpha+1)}(\cosh x)}{P_m^{(-\frac{1}{2},\alpha)}(\cosh x)}. \]
Both the undeformed and the deformed potentials are even functions, and consequently the corresponding Hamiltonians leave invariant the spaces of odd and even functions. The Darboux transformation changes parity. In particular, the deformed even sector is the \( A \)-image of the undeformed odd sector.

We now determine explicitly a basis of the invariant flag corresponding to the even sector of the first and second deformation. To do so, we switch to the algebraic variable and perform a change of gauge so that the undeformed, odd algebraic sector is isomorphic to the standard polynomial flag (c.f. Case IIIa of Table I).

\[ T = z(1-z)\partial_{zz} + ((\alpha - \frac{5}{2})z + 1 - \alpha)\partial_z - \left( \frac{\alpha}{2} - \frac{3}{4} \right)^2, \]
\[ T = e^{-\rho}H_{PT}e^{\rho}, \quad e^{\rho} = \sinh(\frac{\alpha}{2}) \cosh(\frac{\alpha}{2})^{\frac{1}{2} - \alpha}, \]
\[ z = \cosh^2(\frac{\alpha}{2}) = \frac{1}{2}(\cosh x + 1). \]

In the algebraic gauge, the factorization functions for the backward transformations are given by
\[ \phi = e^{-\rho}\phi^{(m)}_{PT} = z^\alpha (z - 1)^{\frac{1}{2}} P_m^{(-\frac{1}{2},\alpha)}(2z - 1) \]
This factorization function is of algebraic type with
\[ (\log \phi)_z = \frac{\alpha}{z} + \frac{1}{2} \frac{1}{1 - z} + \frac{1}{2} + \alpha + m \frac{P_{m-1}^{(\frac{1}{2},\alpha+1)}(2z - 1)}{P_m^{(-\frac{1}{2},\alpha)}(2z - 1)} = \frac{a}{b}, \]
\[ a = ((\frac{1}{2} - \alpha)z + \alpha)P_m^{(-\frac{1}{2},\alpha)}(2z - 1) + \frac{1}{2} + \alpha + m)z(1 - z)P_{m-1}^{(\frac{1}{2},\alpha+1)}(2z - 1), \]
\[ b = z(1 - z)P_m^{(-\frac{1}{2},\alpha)}(2z - 1). \]
A direct calculation shows that the above functions satisfy condition [28], and therefore \( \phi = 1 \). The operators
\[ A = b\partial_z - a, \quad B = g^{-1}\partial_z, \]
where
\[ g = P_m^{(-\frac{1}{2},\alpha)}(2z - 1), \]
constitute a deformation pair of order \(m\). Let us consider the cases \(m = 1\) and \(m = 2\) in more detail. For \(m = 1\) we have
\[
A = \frac{z_1(1 - z_1)(z_1 + 2(\alpha + 1))}{2(2\alpha + 3)} \partial_{z_1} + \frac{(2\alpha + 1)z_1^2 - 4(\alpha + 1)}{4(2\alpha + 3)}
\]
\[
B = \frac{2}{z_1} \partial_{z_1},
\]
\[
\hat{T} = z(1 - z)\partial_{z z} + \left(\frac{z_1}{2} - \frac{4(\alpha + 1)}{(2\alpha + 3)z_1} + \frac{2\alpha + 1}{2\alpha + 3}\right) \partial_{z}
\]
\[
z_1 = 2P_1^{(-\frac{1}{2},\alpha)}(2z - 1) = (2\alpha + 3)z - 2\alpha - 2
\]
The operator \(A = b(z_1)\partial_{z_1} - a(z_1)\), relative to the \(z_1\) variable, is of the form \(3\), and hence, the partner operator \(\hat{T}\) is P.E.S. with invariant subspaces
\[
\hat{\mathcal{M}}_{n+1} = \langle 1, z_2^2, z_1^3, \ldots, z_1^n \rangle.
\]
For \(m = 2\) we set
\[
z_2 = \frac{1}{2}\beta \left((2\alpha + 7)z - (2\alpha + 4)\right),
\]
so that
\[
P_2^{(-\frac{1}{2},\alpha)}(2z - 1) = \frac{3(\alpha + 2)}{2(2\alpha + 7)} (z_2^2 - 1).
\]
Consequently,
\[
A = \left(\frac{-6\beta z_2^2}{2\alpha + 5} - \frac{3(2\alpha + 1)z_2}{3\alpha + 6} + \frac{3\beta}{3\alpha + 6}\right) \frac{3(\alpha + 2)^2(z_2^2 - 1)}{2(2\alpha + 7)^2} \partial_{z_2}
\]
\[
+ \frac{3(\alpha + 2)}{2(2\alpha + 7)^2} \left(-\frac{2\alpha + 3}{2\beta} (z_2^3 - 3z_2) - 2\alpha - 1\right),
\]
\[
B = \frac{2(2\alpha + 7)}{3(\alpha + 2)(z_2^2 - 1)} \partial_{z},
\]
\[
\hat{T} = z(1 - z)\partial_{z z} + \left[\frac{3\beta(\alpha + 2)}{2\alpha + 5} - \frac{4(2\alpha + 3)}{2\alpha + 7} \frac{z_2}{z_2^2 - 1}\right] + \frac{2(2\alpha + 1)(z_2^2 + 1)}{(2\alpha + 7)(z_2^2 - 1)} \partial_{z}
\]
In the same manner, it can be seen from \(3\) that the partner operator \(\hat{T}\) is P.E.S. with invariant subspaces (c.f. \(3\))
\[
\hat{\mathcal{M}}_n = \langle 1, \pi_3(z_2), \pi_4(z_2), \ldots, \pi_{n+1}(z_2) \rangle.
\]

4. Discussion

In this paper we have analyzed the connection between the Darboux transformations and exact solvability by polynomials, i.e. the fact that a certain Hamiltonian operator after a change of variables and a gauge transformation admits an infinite flag of invariant polynomial subspaces. Since operator composition and factorization are covariant with respect to changes of variables and gauge transformations, concepts like the Darboux transformation or shape-invariance are also covariant. It is customary for physical applications to work in the Schrödinger gauge and the physical variable \(x\), but for the purposes of analyzing invariant flags it is more convenient to work in the algebraic variable \(z\) and in the algebraic gauge, the one in which the operator has polynomial eigenfunctions.
A general backwards transformation on a shape invariant potential will lead to a transformed potential whose eigenfunctions are no longer elementary functions. We have discussed the special class of algebraic Darboux transformations of shape-invariant potentials, i.e. those that preserve the exact solvability by polynomials, showing also how the polynomial flag is deformed by the action of the Darboux transformation.

In this paper we placed our emphasis on the action of the Darboux transformation on the invariant flag of subspaces, rather than on the potential. In fact, many different potentials have the same invariant flag, e.g. all the shape-invariant potentials preserve the standard polynomial flag. We have analyzed the deformations of the Pöschl-Teller potential, but similar deformations exist for other shape-invariant forms.

Acknowledgments

The research of DGU is supported in part by a CRM-ISM Postdoctoral Fellowship and the Spanish Ministry of Education under grant EX2002-0176. The research of NK and RM is supported by the Natural Sciences and Engineering Research Council of Canada. NK and DGU would like to acknowledge partial financial support from the project BFM2002-02646 of the Dirección General de Investigación.