Solving the BKP equation via Monte Carlo integration

Grigorios Chachamis, IFT UAM-CSIC Madrid

In collaboration with A. Sabio Vera

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Outline

• Introduction
• The Reggeon
• The Pomeron - BFKL equation
• The Odderon - BKP equation
• Iterative numerical solutions
• Results
• Outlook
Intro: Setting up the stage

- Perturbative QCD
- High energy scattering
- BFKL equation (Balitsky-Fadin-Kuraev-Lipatov)
- BKP equation (Bartels-Kwiecinski-Praszalowicz)
- Pomeron (Named after Pomeranchuk)
- Odderon (Lukaszuk & Nicolescu)
- Reggeon
Intro: The protagonists

Reggeon

Pomeron  Odderon
Intro: Importance of high energy QCD

- High energy QCD studies only a part of the phase space, a certain limit, the limit of scattering at very high energies
- There is a plethora of things though to be learnt from studying that limit, to mention but a few:
  - Integrability
  - Gravity
  - AdS/CFT
  - BDS amplitudes
  - Factorization
  - Separation between transverse and longitudinal d.o.f.
  - Transition from hard to soft scale physics
- And this is only from the 'pure' theory point of view
Intro: Importance of high energy QCD

Rich phenomenology, e.g.

- Multijets
- Rapidity gaps
- DIS
Intro: Partonic vs Hadronic cross-section

**Collinear factorization scheme**
In a hadronic collider things are complicated, one needs to consider the partonic cross-section and convolute that with the PDF’s in order to produce theoretical estimates for an observable.
Intro: Partonic vs Hadronic cross-section

For this talk, all the fun is contained in the Black (Orange) box
Intro: Higher order corrections (schematically)

In which case the perturbative expansion breaks down?
What happens in high energy scattering?
How is the partonic part of the cross section modified?
Intro: Some considerations

• Q: What is the most relevant scale in high energy scattering?
  A: The center-of-mass energy squared $s$

• Q: In which functional form does $s$ appear in the Feynman diagrams?
  A: $\alpha_s^m \ln(s)^n$

• Q: Can one isolate those Feynman diagrams that come with a numerically important $[\alpha_s^m \ln(s)^n \sim 1]$ contribution?
  A: It depends (for this talk the answer is yes)

• Q: Can one resum all these diagrams with important $\alpha_s^m \ln(s)^n$ contributions to all orders in $\alpha_s$?
  A: It depends (for this talk the answer is yes)
The Reggeon
(a reggegeized gluon)

A normal gluon propagator: \[ D_{\mu\nu}(s, q^2) = -i \frac{g_{\mu\nu}}{q^2} \]

All the **virtual corrections** that carry leading-logs in \( s \) are accounted for

The reggeized gluon is a gluon with modified propagator:

\[ D_{\mu\nu}(s, q^2) = -i \frac{g_{\mu\nu}}{q^2} \left( \frac{s}{k^2} \right)^\omega(q^2) \]

From now on, vertical propagators represent Reggeons
Ladder diagrams with two Reggeons

All-orders resummation of $\alpha_s(Q^2) \log \left(\frac{s}{Q^2}\right)$ terms: How? Ladder structure

Loop diagrams can be seen as phase-space diagrams and the reverse

$$A_{\text{elastic}}(s, t) = \sum_n$$

Optical Theorem:

$$\sigma_{\text{TOT}} \approx \frac{1}{s} \text{Im} A_{\text{elastic}}(s, t = 0) = \frac{1}{s} \sum_n$$

$$= \frac{1}{s} \sum_n |A_n(s, t)|^2$$
The Pomeron
(ladder diagrams)

Resum to all $n$?
Use BFKL
The Pomeron (ladder diagrams)

Many many phase-space diagrams

BFKL

GGF
The gluon Green’s function
BFKL equation

What to keep from this figure:
Solving the BFKL equation iteratively amounts to adding one rung with each new iteration
Solve BFKL via Monte Carlo

- Many people have worked on it, the origin goes back to the late 90's:

Effective Feynman Rules:
simplest case, \( t = 0 \), leading order

Gluon Regge trajectory:
\[
\omega (\vec{q}) = -\frac{\alpha_s N_c}{\pi} \log \frac{q^2}{\lambda^2}
\]

Modified \( t \)-channel propagators:
\[
\left( \frac{s_i}{s_0} \right)^{\omega(t_i)} = e^{\omega(t_i)(y_i-y_{i+1})}
\]

\[
\left( \frac{\alpha_s N_c}{\pi} \right)^2 \int d^2 \vec{k}_1 \frac{\theta (k_1^2 - \lambda^2)}{\pi k_1^2} \int d^2 \vec{k}_2 \frac{\theta (k_2^2 - \lambda^2)}{\pi k_2^2} \delta^{(2)} (\vec{k}_A + \vec{k}_1 + \vec{k}_2 - \vec{k}_B) \\
\times \int_0^Y dy_1 \int_0^{y_1} dy_2 e^{\omega(\vec{k}_A)(y-y_1)} e^{\omega(\vec{k}_A+\vec{k}_1)(y_1-y_2)} e^{\omega(\vec{k}_A+\vec{k}_1+\vec{k}_2)y_2}
\]

Very simplified pictorial view, the main elements and ideas are here though
Let us iterate

Vertical lines are Reggeons
horizontal ones are gluons
A few words on color

For QCD, the possible states are:
\[1, 8_A, 8_S, 10 + 10, 27\]

with color factors:
\[-3, -\frac{3}{2}, -\frac{3}{2}, 0, 1\]

\[K_{BFKL}(l, q - l; k, q - k) = -N_c g^2 \left[ q^2 - \frac{k^2(q - l)^2}{(k - l)^2} - \frac{(q - k)^2l^2}{(k - l)^2} \right] + (2\pi)^3 k^2(q - k)^2 [\omega(k) + \omega(q - k)] \delta^{(2)}(k - l).\]

Where \[\omega(k^2) = -\frac{N_c}{2} g^2 \int \frac{d^2l}{(2\pi)^3} \frac{k^2}{l^2(l - k)^2}\]
Color does matter

**Symmetric octet**

- It was in a generalized leading logarithmic approximation, and by iterating the BFKL kernel in the s-channel, where the Bartels-Kwiecinski-Praszalowicz (BKP) equation was proposed
  
  Bartels (1980)
  
  Kwiecinski, Praszalowicz (1980)

- BKP was found to have a hidden integrability being equivalent to a periodic spin chain of a XXX Heisenberg ferromagnet. This was the first example of the existence of integrable systems in QCD
  

- *It will be directly connected to any numerical solution of the BKP, if any such work is to be done with the aim to perform phenomenological studies for the Odderon*

**Antisymmetric octet**

- Corrections to the Bern-Dixon-Smirnov (BDS) iterative ansatz (Bern, Dixon, Smirnov, 2005) for the n-point maximally helicity violating (MHV) and planar amplitudes were found in MRK in the six-point amplitude at two loops
  

In other words, it is a fundamental ingredient of the finite remainder of scattering amplitudes with arbitrary number of external legs and internal loops
The Odderon

- Pomeron is the state of two interacting reggeized gluons in the t-channel in the color singlet. It has the quantum numbers of the vacuum.
- Odderon is the state of three interacting gluons exchanged in the t-channel in the color singlet but with $C = -1$ and $P = -1$.
- Any pair of two gluons in the Odderon forms symmetric color octet subsystems.

Ladder structure of the Odderon. BKP resums term of the form $\alpha_s(\alpha_s \log s)^n$.

NLO corrections recently available

Bartels, Fadin, Lipatov, Vacca (2012)

The Odderon is nowhere to be seen so far.*

* see arXiv:1711.03288
Let us iterate

Vertical lines are Reggeons
horizontal ones are gluons
Let us iterate
Let us iterate

BFKL

Iteration

BKP

Iteration

+ +
\[ O(k) \otimes f(p_1, p_2, p_3) \equiv \xi(p_1, p_2, p_3, k) f(p_1 + k, p_2 - k, p_3) + \xi(p_2, p_3, p_1, k) f(p_1, p_2 + k, p_3 - k) + \xi(p_1, p_3, p_2, k) f(p_1 + k, p_2, p_3 - k) \]
Ternary tree structure

<table>
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<tr>
<th>n rungs</th>
<th>Number of diagrams</th>
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<tr>
<td>2</td>
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</tr>
</tbody>
</table>

first iteration

second iteration
Results

All vectors live in the transverse momentum space

$q = (4, 0)$
$p_1 = (10, 0)$
$p_2 = (20, \pi)$
$p_3 = (q - p_1) - p_2 = (14, 0)$
$p_4 = (20, 0)$
$p_5 = (25, \pi)$
$p_6 = (q - p_4) - p_5 = (9, 0)$

$q = (31, 0)$
$p_1 = (10, 0)$
$p_2 = (20, \pi)$
$p_3 = (q - p_1) - p_2 = (41, 0)$
$p_4 = (20, 0)$
$p_5 = (25, \pi)$
$p_6 = (q - p_4) - p_5 = (36, 0)$

$(r, \theta)$: first component is in GeV, the second component in radians
Results

\[
\frac{df}{dn} = \begin{cases} 
1 & Y = 1 \\
1.5 & Y = 1.5 \\
2 & Y = 2 \\
2.5 & Y = 2.5 \\
3 & Y = 3 \\
3.5 & Y = 3.5 \\
4 & Y = 4 \\
4.5 & Y = 4.5 \\
5 & Y = 5 \\
5.5 & Y = 5.5 
\end{cases}
\]
Results

\[ \frac{df}{dn} \]

Number or rungs \( n \)

\[ q = 31 \text{ GeV} \]

\[ Y = 1 \]
\[ Y = 1.5 \]
\[ Y = 2 \]
\[ Y = 2.5 \]
\[ Y = 3 \]

\[ \frac{df}{dn} \]

Number or rungs \( n \)

\[ q = 31 \text{ GeV} \]

\[ Y = 3.5 \]
\[ Y = 4 \]
\[ Y = 4.5 \]
\[ Y = 5 \]
\[ Y = 5.5 \]

\[ \frac{df}{dn} \]

Number of rungs \( n \)

\[ Y \]

\[ \frac{df}{dn} \]

Number of rungs \( n \)

\[ Y \]
Conclusions and Outlook

• We have used a Monte Carlo numerical integration of iterated integrals in transverse momentum and rapidity space to solve the BKP equation with three Reggeized gluons in the t-channel (Odderon case).

• Numerical convergence of the solution is achieved after applying the BKP ternary kernel on the initial condition, corresponding to three off-shell gluon propagators, a finite number of times for a given value of the strong coupling and the center-of-mass energy (in terms of rapidity, Y).

• The gluon Green function for Reggeized gluons grows with Y for small values of this variable to then decrease at higher Y.

• The formalism can be applied to the BKP equation with a higher number of exchanged Reggeons. It can also be used beyond the leading logarithmic approximation and for cases with a total t-channel color projection not being in the singlet but in the adjoint representation. This is very important for the calculation of scattering amplitudes in $N = 4$ supersymmetric theories in the Regge limit. (G.C. and A. Sabio Vera, work in progress)

• Our approach also has obvious applications in the study of phenomenological cross sections devoted to the search of the elusive Odderon at hadron colliders.